LECTURE 4

LECTURE OUTLINE

- Review of relative interior
- Algebra of relative interiors and closures
- Continuity of convex functions
- Recession cones

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- Recall: x is a *relative interior point* of C, if x is an interior point of C relative to $\text{aff}(C)$
- Three fundamental properties of $\text{ri}(C)$ of a convex set C:
  - $\text{ri}(C)$ is nonempty
  - If $x \in \text{ri}(C)$ and $\overline{x} \in \text{cl}(C)$, then all points on the line segment connecting x and $\overline{x}$, except possibly $\overline{x}$, belong to $\text{ri}(C)$
  - If $x \in \text{ri}(C)$ and $\overline{x} \in C$, the line segment connecting $\overline{x}$ and x can be prolonged beyond x without leaving C

CLOSURE VS RELATIVE INTERIOR

- Let C be a nonempty convex set. Then $\mathrm{ri}(C)$ and $\mathrm{cl}(C)$ are "not too different for each other."

- *Proposition*:

  (a) We have $\mathrm{cl}(C) = \mathrm{cl}\big(\mathrm{ri}(C)\big)$.

  (b) We have $\mathrm{ri}(C) = \mathrm{ri}\big(\mathrm{cl}(C)\big)$.

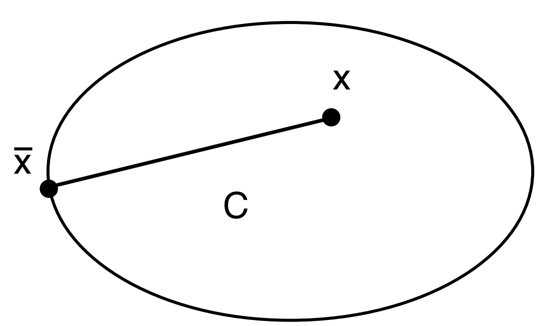
  (c) Let $\overline{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

    (i) C and $\overline{C}$ have the same rel. interior.

    (ii) C and $\overline{C}$ have the same closure.

    (iii) $\mathrm{ri}(C) \subset \overline{C} \subset \mathrm{cl}(C)$.

Proof: (a) Since $\mathrm{ri}(C) \subset C$, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(C)$. Conversely, let $\overline{x} \in \mathrm{cl}(C)$. Let $x \in \mathrm{ri}(C)$. By the Line Segment Principle, we have $\alpha x + (1-\alpha)\overline{x} \in \mathrm{ri}(C)$ for all $\alpha \in (0, 1]$. Thus, $\overline{x}$ is the limit of a sequence that lies in $\mathrm{ri}(C)$, so $\overline{x} \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$.



ALGEBRA OF CLOSURES AND REL. INTERIORS I

- Let C be a nonempty convex subset of $\Re^n$ and let A be an $m \times n$ matrix.

  (a) We have $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$.

  (b) We have $A \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A \cdot C)$. Furthermore, if C is bounded, then $A \cdot \mathrm{cl}(C) = \mathrm{cl}(A \cdot C)$.

Proof: (b) We have $A \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A \cdot C)$, since if a sequence $\{x_k\} \subset C$ converges to some $x \in \mathrm{cl}(C)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot C$, converges to Ax, implying that $Ax \in \mathrm{cl}(A \cdot C)$.

To show the converse, assuming that C is bounded, choose any $z \in \mathrm{cl}(A \cdot C)$. Then, there exists a sequence $\{x_k\} \subset C$ such that $Ax_k \to z$. Since C is bounded, $\{x_k\}$ has a subsequence that converges to some $x \in \mathrm{cl}(C)$, and we must have $Ax = z$. It follows that $z \in A \cdot \mathrm{cl}(C)$. **Q.E.D.**

Note that in general, we may have

$$A \cdot \mathrm{int}(C) \neq \mathrm{int}(A \cdot C)$$

$$A \cdot \mathrm{cl}(C) \neq \mathrm{cl}(A \cdot C)$$

ALGEBRA OF CLOSURES AND REL. INTERIORS II

- Let C_1 and C_2 be nonempty convex sets.

(a) If $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, then

$$\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2),$$

$$\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2).$$

(b) We have

$$\mathrm{ri}(C_1 + C_2) = \mathrm{ri}(C_1) + \mathrm{ri}(C_2),$$

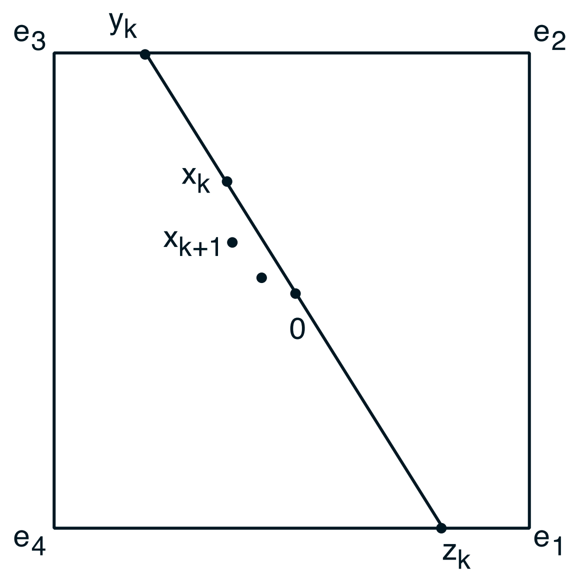
$$\mathrm{cl}(C_1) + \mathrm{cl}(C_2) \subset \mathrm{cl}(C_1 + C_2).$$

If one of C_1 and C_2 is bounded, then

$$\mathrm{cl}(C_1) + \mathrm{cl}(C_2) = \mathrm{cl}(C_1 + C_2).$$

CONTINUITY OF CONVEX FUNCTIONS

- If $f : \Re^n \mapsto \Re$ is convex, then it is continuous.



Proof: We will show that f is continuous at 0. By convexity, f is bounded within the unit cube by the maximum value of f over the corners of the cube.

Consider sequence $x_k \to 0$ and the sequences $y_k = x_k/\|x_k\|_\infty$, $z_k = -x_k/\|x_k\|_\infty$. Then

$$f(x_k) \leq \big(1 - \|x_k\|_\infty\big) f(0) + \|x_k\|_\infty f(y_k)$$

$$f(0) \leq \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) + \frac{1}{\|x_k\|_\infty + 1} f(x_k)$$

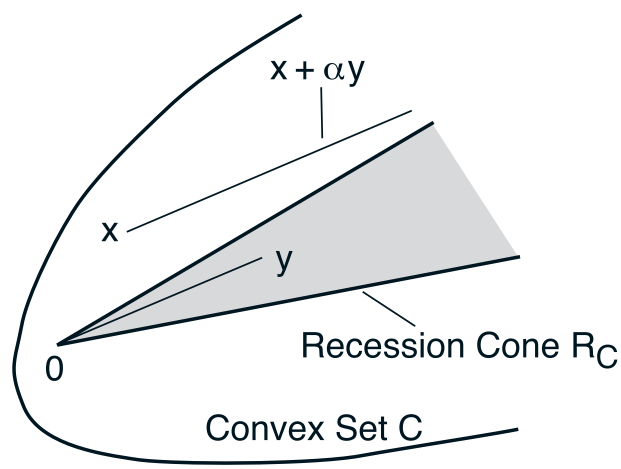
Since $\|x_k\|_\infty \to 0$, $f(x_k) \to f(0)$. **Q.E.D.**

- Extension to continuity over $\mathrm{ri}(\mathrm{dom}(f))$.

RECESSION CONE OF A CONVEX SET

- Given a nonempty convex set C, a vector y is a *direction of recession* if starting at any x in C and going indefinitely along y, we never cross the relative boundary of C to points outside C:

$$x + \alpha y \in C, \qquad \forall\, x \in C,\ \ \forall\, \alpha \geq 0$$



- *Recession cone* of C (denoted by R_C): The set of all directions of recession.

- R_C is a cone containing the origin.

RECESSION CONE THEOREM

- Let C be a nonempty closed convex set.

  (a) The recession cone R_C is a closed convex cone.

  (b) A vector y belongs to R_C if and only if there exists a vector $x \in C$ such that $x + \alpha y \in C$ for all $\alpha \geq 0$.

  (c) R_C contains a nonzero direction if and only if C is unbounded.

  (d) The recession cones of C and $\mathrm{ri}(C)$ are equal.

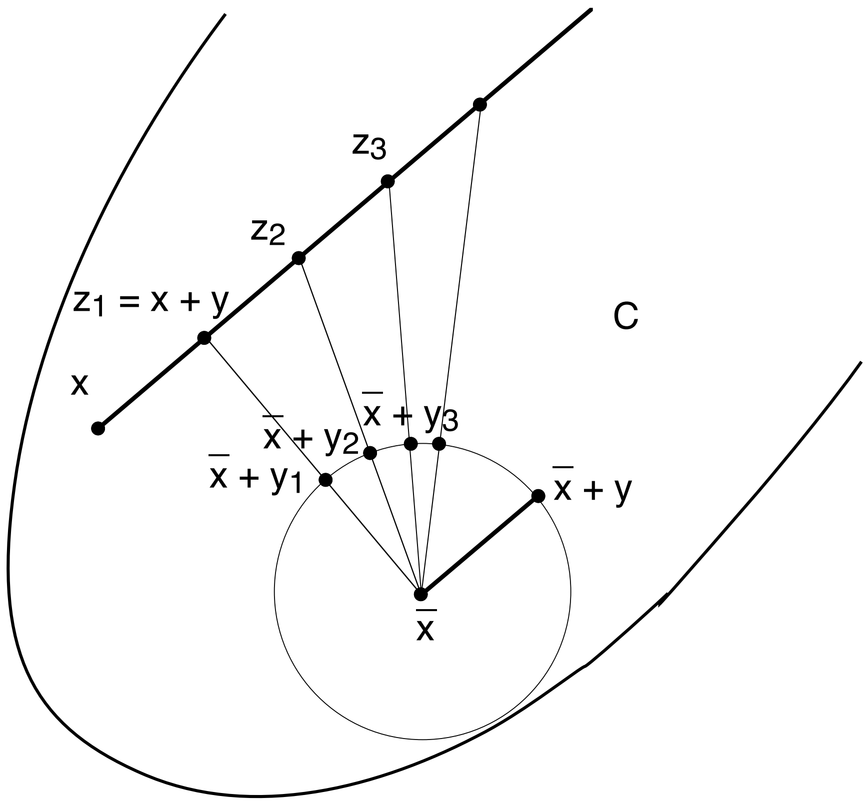
  (e) If D is another closed convex set such that $C \cap D \neq \emptyset$, we have

  $$R_{C \cap D} = R_C \cap R_D.$$

  More generally, for any collection of closed convex sets C_i, $i \in I$, where I is an arbitrary index set and $\cap_{i \in I} C_i$ is nonempty, we have

  $$R_{\cap_{i \in I} C_i} = \cap_{i \in I} R_{C_i}.$$

PROOF OF PART (B)



- Let $y \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha y \in C$ for all $\alpha \geq 0$. We fix $\overline{x} \in C$ and $\alpha > 0$, and we show that $\overline{x} + \alpha y \in C$. By scaling y, it is enough to show that $\overline{x} + y \in C$.

  Let $z_k = x + ky$ for $k = 1, 2, \ldots$, and $y_k = (z_k - \overline{x})\|y\|/\|z_k - \overline{x}\|$. We have

$$\frac{y_k}{\|y\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|}\frac{y}{\|y\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|}, \quad \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \to 1, \quad \frac{x - \overline{x}}{\|z_k - \overline{x}\|} \to 0,$$

so $y_k \to y$ and $\overline{x} + y_k \to \overline{x} + y$. Use the convexity and closedness of C to conclude that $\overline{x} + y \in C$.

LINEALITY SPACE

- The *lineality space* of a convex set C, denoted by L_C, is the subspace of vectors y such that $y \in R_C$ and $-y \in R_C$:

$$L_C = R_C \cap (-R_C).$$

- *Decomposition of a Convex Set*: Let C be a nonempty convex subset of $\Re^n$. Then, for every subspace S that is contained in the lineality space L_C, we have

$$C = S + (C \cap S^\perp).$$