LECTURE 9

LECTURE OUTLINE

- Min-Max Problems
- Saddle Points
- Min Common/Max Crossing for Min-Max

Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$ consider

$$\text{minimize} \ \sup_{z \in Z} \phi(x, z)$$
$$\text{subject to} \ x \in X$$

and

$$\text{maximize} \ \inf_{x \in X} \phi(x, z)$$
$$\text{subject to} \ z \in Z.$$

- Minimax inequality (holds always)

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

SADDLE POINTS

Definition: (x^*, z^*) is called a *saddle point* of ϕ if

$$\phi(x^*, z) \le \phi(x^*, z^*) \le \phi(x, z^*), \quad \forall\, x \in X, \forall\, z \in Z$$

Proposition: (x^*, z^*) is a saddle point if and only if the minimax equality holds and

$$x^* \in \arg\min_{x \in X} \sup_{z \in Z} \phi(x, z), \quad z^* \in \arg\max_{z \in Z} \inf_{x \in X} \phi(x, z) \quad (*)$$

Proof: If (x^*, z^*) is a saddle point, then

$$\inf_{x \in X} \sup_{z \in Z} \phi(x, z) \le \sup_{z \in Z} \phi(x^*, z) = \phi(x^*, z^*)$$

$$= \inf_{x \in X} \phi(x, z^*) \le \sup_{z \in Z} \inf_{x \in X} \phi(x, z)$$

By the minimax inequality, the above holds as an equality holds throughout, so the minimax equality and Eq. (*) hold.

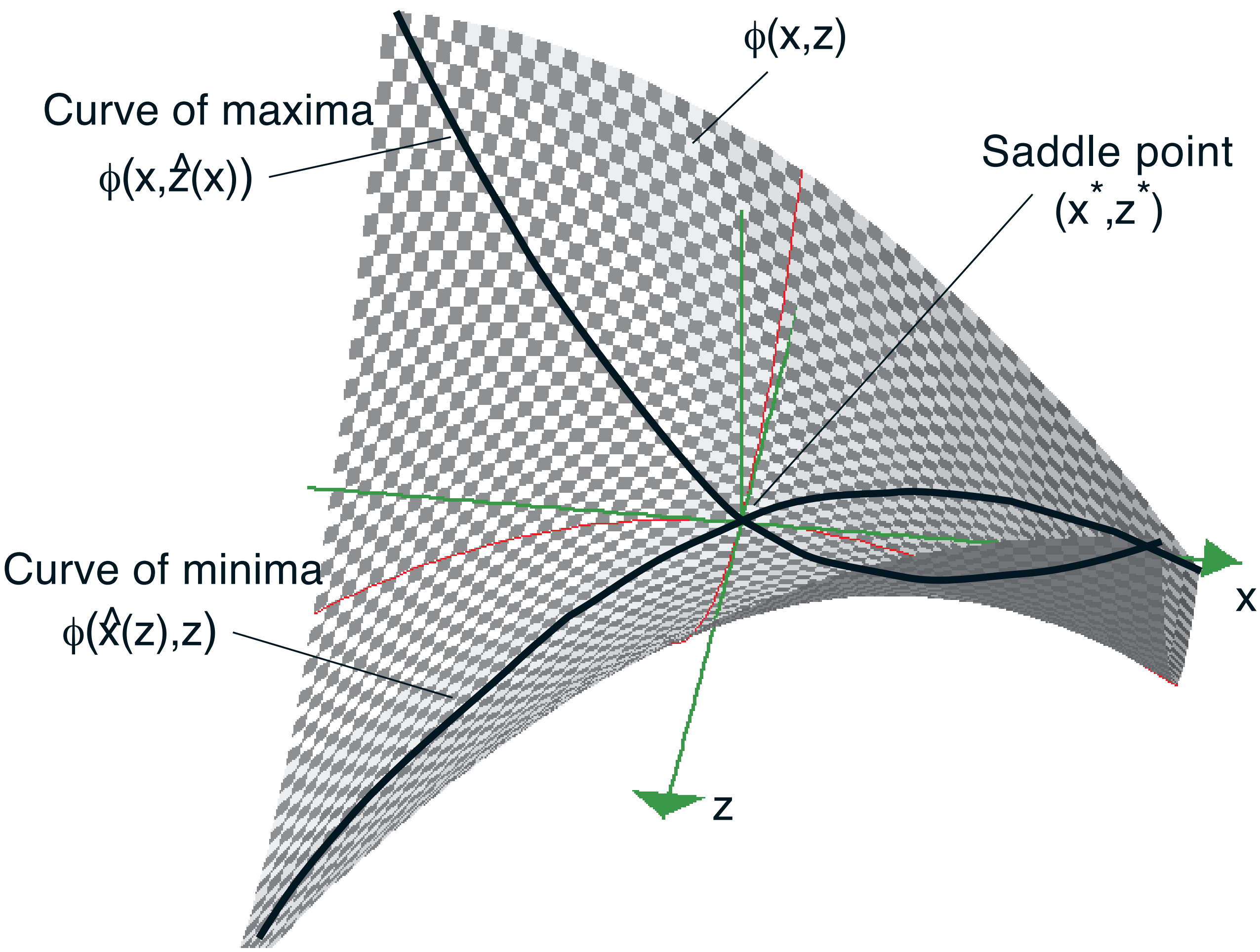
Conversely, if Eq. (*) holds, then

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \phi(x, z^*) \le \phi(x^*, z^*)$$

$$\le \sup_{z \in Z} \phi(x^*, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

Using the minimax equ., (x^*, z^*) is a saddle point.

VISUALIZATION



The curve of maxima $\phi(x, \hat{z}(x))$ lies above the curve of minima $\phi(\hat{x}(z), z)$, where

$$\hat{z}(x) = \arg\max_z \phi(x, z), \qquad \hat{x}(z) = \arg\min_x \phi(x, z).$$

Saddle points correspond to points where these two curves meet.

MIN COMMON/MAX CROSSING FRAMEWORK

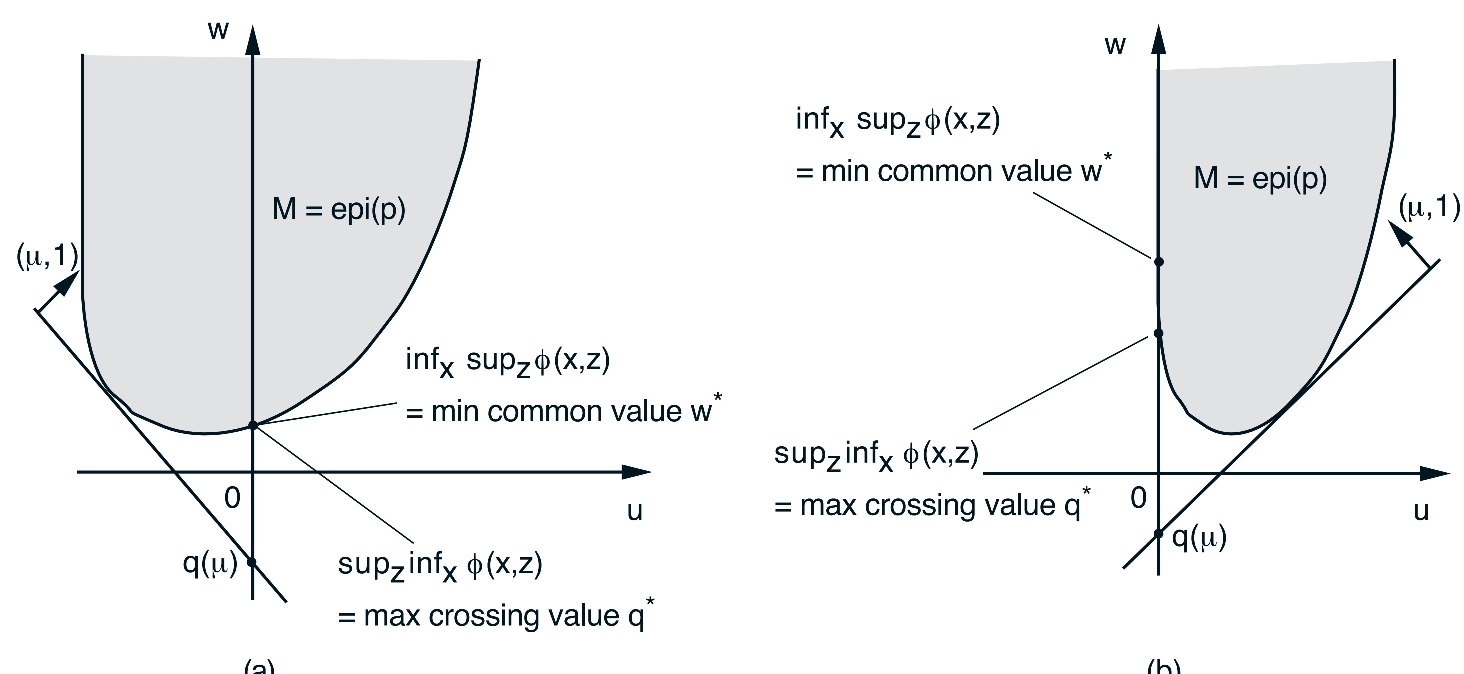
- Introduce perturbation function $p : \Re^m \mapsto [-\infty, \infty]$

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}, \qquad u \in \Re^m$$

- Apply the min common/max crossing framework with the set M equal to the epigraph of p.

- Application of a more general idea: To evaluate a quantity of interest w^*, introduce a suitable perturbation u and function p, with $p(0) = w^*$.

- Note that $w^* = \inf \sup \phi$. We will show that:
  - Convexity in x implies that M is a convex set.
  - Concavity in z implies that $q^* = \sup \inf \phi$.



IMPLICATIONS OF CONVEXITY IN X

Lemma 1: Assume that X is convex and that for each $z \in Z$, the function $\phi(\cdot, z) : X \mapsto \Re$ is convex. Then p is a convex function.

Proof: Let

$$F(x,u) = \begin{cases} \sup_{z \in Z} \{\phi(x,z) - u'z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

Since $\phi(\cdot, z)$ is convex, and taking pointwise supremum preserves convexity, F is convex. Since

$$p(u) = \inf_{x \in \Re^n} F(x,u),$$

and partial minimization preserves convexity, the convexity of p follows from the convexity of F. **Q.E.D.**

THE MAX CROSSING PROBLEM

- The max crossing problem is to maximize $q(\mu)$ over $\mu \in \Re^n$, where

$$q(\mu) = \inf_{(u,w)\in \mathrm{epi}(p)} \{w + \mu' u\} = \inf_{\{(u,w)\mid p(u)\le w\}} \{w + \mu' u\}$$
$$= \inf_{u\in\Re^m} \big\{p(u) + \mu' u\big\}$$

Using $p(u) = \inf_{x\in X} \sup_{z\in Z} \big\{\phi(x,z) - u'z\big\}$, we obtain

$$q(\mu) = \inf_{u\in\Re^m} \inf_{x\in X} \sup_{z\in Z} \big\{\phi(x,z) + u'(\mu - z)\big\}$$

- By setting $z = \mu$ in the right-hand side,

$$\inf_{x\in X} \phi(x,\mu) \le q(\mu), \qquad \forall\, \mu \in Z.$$

Hence, using also weak duality ($q^* \le w^*$),

$$\sup_{z\in Z} \inf_{x\in X} \phi(x,z) \le \sup_{\mu\in\Re^m} q(\mu) = q^*$$
$$\le w^* = p(0) = \inf_{x\in X} \sup_{z\in Z} \phi(x,z)$$

IMPLICATIONS OF CONCAVITY IN Z

Lemma 2: Assume that for each $x \in X$, the function $r_x : \Re^m \mapsto (-\infty, \infty]$ defined by

$$r_x(z) = \begin{cases} -\phi(x, z) & \text{if } z \in Z, \\ \infty & \text{otherwise,} \end{cases}$$

is closed and convex. Then

$$q(\mu) = \begin{cases} \inf_{x \in X} \phi(x, \mu) & \text{if } \mu \in Z, \\ -\infty & \text{if } \mu \notin Z. \end{cases}$$

Proof: (Outline) From the preceding slide,

$$\inf_{x \in X} \phi(x, \mu) \leq q(\mu), \qquad \forall\, \mu \in Z.$$

We show that $q(\mu) \leq \inf_{x \in X} \phi(x, \mu)$ for all $\mu \in Z$ and $q(\mu) = -\infty$ for all $\mu \notin Z$, by considering separately the two cases where $\mu \in Z$ and $\mu \notin Z$.

First assume that $\mu \in Z$. Fix $x \in X$, and for $\epsilon > 0$, consider the point $\big(\mu, r_x(\mu) - \epsilon\big)$, which does not belong to epi(r_x). Since epi(r_x) does not contain any vertical lines, there exists a nonvertical strictly separating hyperplane ...

MINIMAX THEOREM I

Assume that:

(1) X and Z are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \Re$ is closed and convex.

Then, the minimax equality holds if and only if the function p is lower semicontinuous at $u = 0$.

Proof: The convexity/concavity assumptions guarantee that the minimax equality is equivalent to $q^* = w^*$ in the min common/max crossing framework. Furthermore, $w^* < \infty$ by assumption, and the set $\overline{M}$ [equal to M and $\mathrm{epi}(p)$] is convex.

By the 1st Min Common/Max Crossing Theorem, we have $w^* = q^*$ iff for every sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k \to \infty} w_k$. This is equivalent to the lower semicontinuity assumption on p:

$$p(0) \leq \liminf_{k \to \infty} p(u_k), \quad \text{for all } \{u_k\} \text{ with } u_k \to 0$$

MINIMAX THEOREM II

Assume that:

(1) X and Z are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) > -\infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \Re$ is closed and convex.

(5) 0 lies in the relative interior of $\mathrm{dom}(p)$.

Then, the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$. [Also the set of z where the sup is attained is compact if 0 is in the interior of $\mathrm{dom}(f)$.]

Proof: Apply the 2nd Min Common/Max Crossing Theorem.

EXAMPLE I

- Let $X = \big\{(x_1, x_2) \mid x \geq 0\big\}$ and $Z = \{z \in \Re \mid z \geq 0\}$, and let

$$\phi(x, z) = e^{-\sqrt{x_1 x_2}} + z x_1,$$

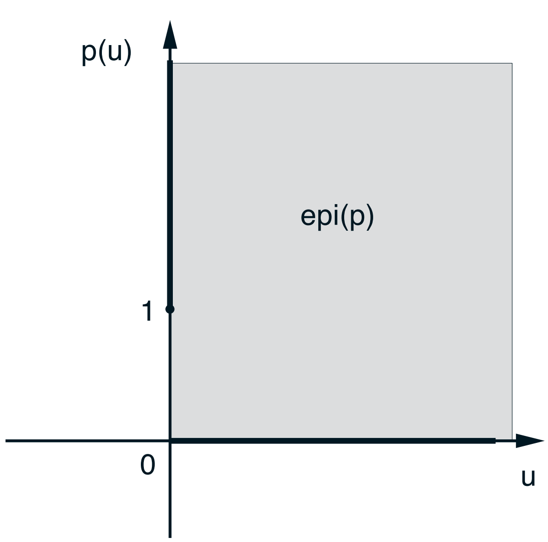
which satisfy the convexity and closedness assumptions. For all $z \geq 0$,

$$\inf_{x \geq 0} \big\{e^{-\sqrt{x_1 x_2}} + z x_1\big\} = 0,$$

so $\sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0$. Also, for all $x \geq 0$,

$$\sup_{z \geq 0} \big\{e^{-\sqrt{x_1 x_2}} + z x_1\big\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0, \end{cases}$$

so $\inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1$.



$$p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \big\{e^{-\sqrt{x_1 x_2}} + z(x_1 - u)\big\}$$

$$= \begin{cases} \infty & \text{if } u < 0, \\ 1 & \text{if } u = 0, \\ 0 & \text{if } u > 0, \end{cases}$$

EXAMPLE II

- Let $X = \Re$, $Z = \{z \in \Re \mid z \geq 0\}$, and let

$$\phi(x, z) = x + zx^2,$$

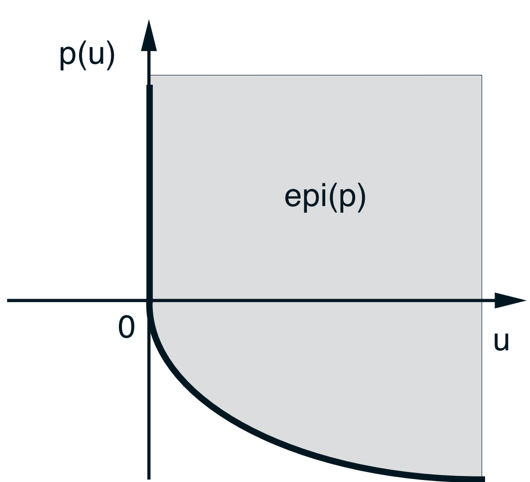
which satisfy the convexity and closedness assumptions. For all $z \geq 0$,

$$\inf_{x \in \Re} \{x + zx^2\} = \begin{cases} -1/(4z) & \text{if } z > 0, \\ -\infty & \text{if } z = 0, \end{cases}$$

so $\sup_{z \geq 0} \inf_{x \in \Re} \phi(x, z) = 0$. Also, for all $x \in \Re$,

$$\sup_{z \geq 0} \{x + zx^2\} = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise}, \end{cases}$$

so $\inf_{x \in \Re} \sup_{z \geq 0} \phi(x, z) = 0$. However, the sup is not attained.



$$p(u) = \inf_{x \in \Re} \sup_{z \geq 0} \{x + zx^2 - uz\}$$
$$= \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases}$$

CONDITIONS FOR ATTAINING THE MIN

Define

$$r_x(z) = \begin{cases} -\phi(x,z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z, \end{cases} \qquad r(z) = \sup_{x \in X} r_x(z)$$

$$t_z(x) = \begin{cases} \phi(x,z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \qquad t(x) = \sup_{z \in Z} t_z(x)$$

Assume that:

(1) X and Z are convex, and t is proper, i.e.,

$$\inf_{x \in X} \sup_{z \in Z} \phi(x,z) < \infty.$$

(2) For each $x \in X$, $r_x(\cdot)$ is closed and convex, and for each $z \in Z$, $t_z(\cdot)$ is closed and convex.

(3) All the level sets $\{x \mid t(x) \leq \gamma\}$ are compact.

Then, the minimax equality holds, and the set of points attaining the inf in $\inf_{x \in X} \sup_{z \in Z} \phi(x,z)$ is nonempty and compact.

• **Note:** Condition (3) can be replaced by more general directions of recession conditions.

PROOF

Note that p is obtained by the partial minimization

$$p(u) = \inf_{x \in \Re^n} F(x, u),$$

where

$$F(x,u) = \begin{cases} \sup_{z \in Z}\big\{\phi(x,z) - u'z\big\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

We have

$$t(x) = F(x, 0),$$

so the compactness assumption on the level sets of t can be translated to the compactness assumption of the partial minimization theorem. It follows from that theorem that p is closed and proper.

By the Minimax Theorem I, using the closedness of p, it follows that the minimax equality holds.

The infimum over X in the right-hand side of the minimax equality is attained at the set of minimizing points of the function t, which is nonempty and compact since t is proper and has compact level sets.

SADDLE POINT THEOREM

Define

$$r_x(z) = \begin{cases} -\phi(x,z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z, \end{cases} \qquad r(z) = \sup_{x \in X} r_x(z)$$

$$t_z(x) = \begin{cases} \phi(x,z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \qquad t(x) = \sup_{z \in Z} t_z(x)$$

Assume that:

(1) X and Z are convex and

either $-\infty < \sup_{z \in Z} \inf_{x \in X} \phi(x,z)$, or $\inf_{x \in X} \sup_{z \in Z} \phi(x,z) < \infty$.

(2) For each $x \in X$, $r_x(\cdot)$ is closed and convex, and for each $z \in Z$, $t_z(\cdot)$ is closed and convex.

(3) All the level sets $\{x \mid t(x) \leq \gamma\}$ and $\{z \mid r(z) \leq \gamma\}$ are compact.

Then, the minimax equality holds, and the set of saddle points of ϕ is nonempty and compact.

Proof: Apply the preceding theorem. **Q.E.D.**

SADDLE POINT COROLLARY

Assume that X and Z are convex, t_z and r_x are closed and convex for all $z \in Z$ and $x \in X$, respectively, and any one of the following holds:

(1) X and Z are compact.

(2) Z is compact and there exists a vector $\overline{z} \in Z$ and a scalar γ such that the level set $\{x \in X \mid \phi(x, \overline{z}) \leq \gamma\}$ is nonempty and compact.

(3) X is compact and there exists a vector $\overline{x} \in X$ and a scalar γ such that the level set $\{z \in Z \mid \phi(\overline{x}, z) \geq \gamma\}$ is nonempty and compact.

(4) There exist vectors $\overline{x} \in X$ and $\overline{z} \in Z$, and a scalar γ such that the level sets

$$\{x \in X \mid \phi(x, \overline{z}) \leq \gamma\}, \quad \{z \in Z \mid \phi(\overline{x}, z) \geq \gamma\},$$

are nonempty and compact.

Then, the minimax equality holds, and the set of saddle points of ϕ is nonempty and compact.