LECTURE 12

LECTURE OUTLINE

- Polyhedral aspects of duality
- Hyperplane proper polyhedral separation
- Min Common/Max Crossing Theorem under polyhedral assumptions
- Nonlinear Farkas Lemma
- Application to convex programming

HYPERPLANE PROPER POLYHEDRAL SEPARATION

- Recall that two convex sets C and P such that

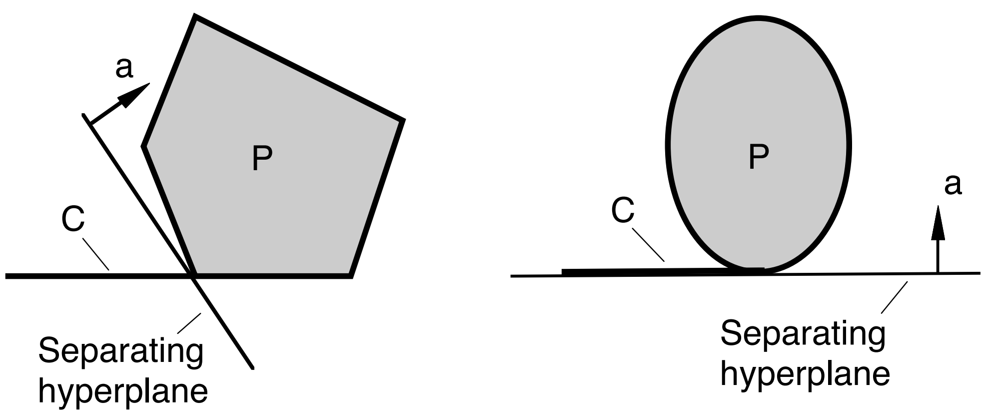
$$\mathrm{ri}(C) \cap \mathrm{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both C and P.

- If P is polyhedral and the slightly stronger condition

$$\mathrm{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the nonpolyhedral set C while it may contain P.



On the left, the separating hyperplane can be chosen so that it does not contain C. On the right where P is not polyhedral, this is not possible.

MIN COMMON/MAX CROSSING TH. - SIMPLE

- Consider the min common and max crossing problems, and assume that:

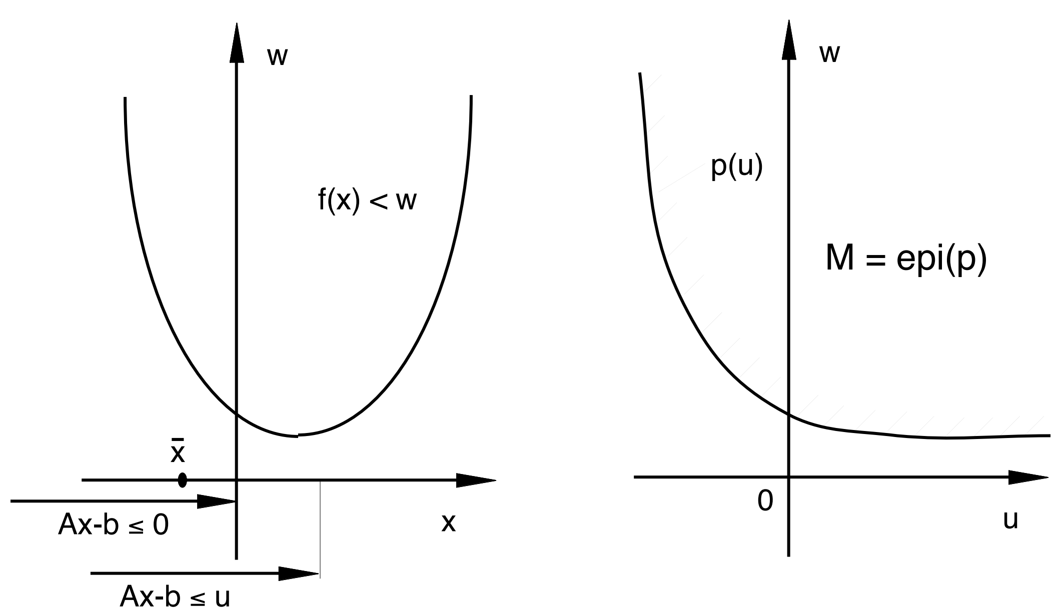
  (1) The set M is defined in terms of a convex function $f : \Re^m \mapsto (-\infty, \infty]$, an $r \times m$ matrix A, and a vector $b \in \Re^r$:

$$M = \Big\{(u, w) \mid \text{for some } (x, w) \in \text{epi}(f),\ Ax - b \le u\Big\}$$

  (2) There is an $\overline{x} \in \text{ri}(\text{dom}(f))$ s. t. $A\overline{x} - b \le 0$.

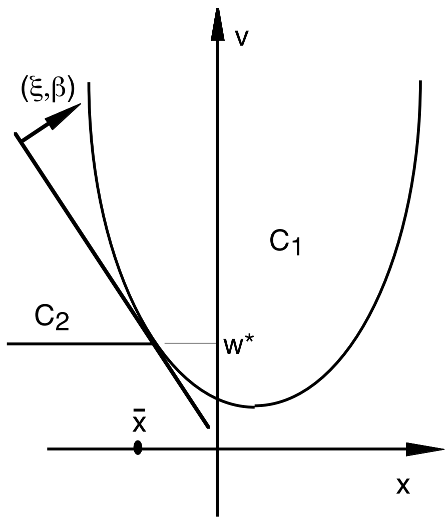
Then $q^* = w^*$ and there is a $\mu \ge 0$ with $q(\mu) = q^*$.

- We have $M \approx \text{epi}(p)$, where $p(u) = \inf_{Ax-b\le u} f(x)$.
- We have $w^* = p(0) = \inf_{Ax-b\le 0} f(x)$.



PROOF

- Consider the disjoint convex sets



$$C_1 = \Big\{(x, v) \mid f(x) < v\Big\}$$

$$C_2 = \Big\{(x, w^*) \mid Ax - b \leq 0\Big\}$$

- Since C_2 is polyhedral, there exists a separating hyperplane not containing C_1, i.e., a $(\xi, \beta) \neq (0, 0)$

$$\beta w^* + \xi' z \leq \beta v + \xi' x, \ \forall\, (x, v) \in C_1, \ \forall\, z \text{ with } Az - b \leq 0,$$

$$\inf_{(x,v)\in C_1} \Big\{\beta v + \xi' x\Big\} < \sup_{(x,v)\in C_1} \Big\{\beta v + \xi' x\Big\}.$$

Because of the relative interior point, $\beta \neq 0$, so we may assume that $\beta = 1$. Hence

$$\sup_{Az-b\leq 0} \Big\{w^* + \xi' z\Big\} \leq \inf_{(x,w)\in \mathrm{epi}(f)} \Big\{w + \xi' x\Big\}.$$

The LP on the left has an optimal solution z^*.

PROOF (CONTINUED)

- Let a_j' be the rows of A, and $\overline{J} = \{j \mid a_j' z^* = b_j\}$. We have

$$\xi' y \leq 0, \qquad \forall\, y \text{ with } a_j' y \leq 0,\ \forall\, j \in \overline{J},$$

so by Farkas' Lemma, there exist $\mu_j \geq 0$, $i \in \overline{J}$, such that $\xi = \sum_{j \in \overline{J}} \mu_j a_j$. Defining $\mu_j = 0$ for $j \notin \overline{J}$, we have

$$\xi = A'\mu \text{ and } \mu'(Az^* - b) = 0, \quad \text{so } \xi' z^* = \mu' b.$$

- Hence from $w^* + \xi' z^* \leq \inf_{(x,w) \in \mathrm{epi}(f)} \bigl\{w + \xi' x\bigr\}$,

$$\begin{aligned}
w^* &\leq \inf_{(x,w) \in \mathrm{epi}(f)} \Bigl\{w + \mu'(Ax - b)\Bigr\} \\
&\leq \inf_{\substack{(x,w) \in \mathrm{epi}(f), \\ Ax - b \leq u}} \{w + \mu'(Ax - b)\} \\
&\leq \inf_{\substack{(x,w) \in \mathrm{epi}(f),\ u \in \Re^n \\ Ax - b \leq u}} \{w + \mu' u\} \\
&= \inf_{(u,w) \in M} \{w + \mu' u\} = q(\mu) \leq q^*.
\end{aligned}$$

Since generically $q^* \leq w^*$, it follows that $q(\mu) = q^* = w^*$. $\quad$ **Q.E.D.**

NONLINEAR FARKAS' LEMMA

- Let $C \subset \Re^n$ be convex, and $f : C \mapsto \Re$ and $g_j : C \mapsto \Re$, $j = 1, \ldots, r$, be convex functions. Assume that

$$f(x) \geq 0, \qquad \forall\, x \in F = \big\{x \in C \mid g(x) \leq 0\big\},$$

and one of the following two conditions holds:

(1) 0 is in the relative interior of the set $D = \big\{u \mid g(x) \leq u \text{ for some } x \in C\big\}$.

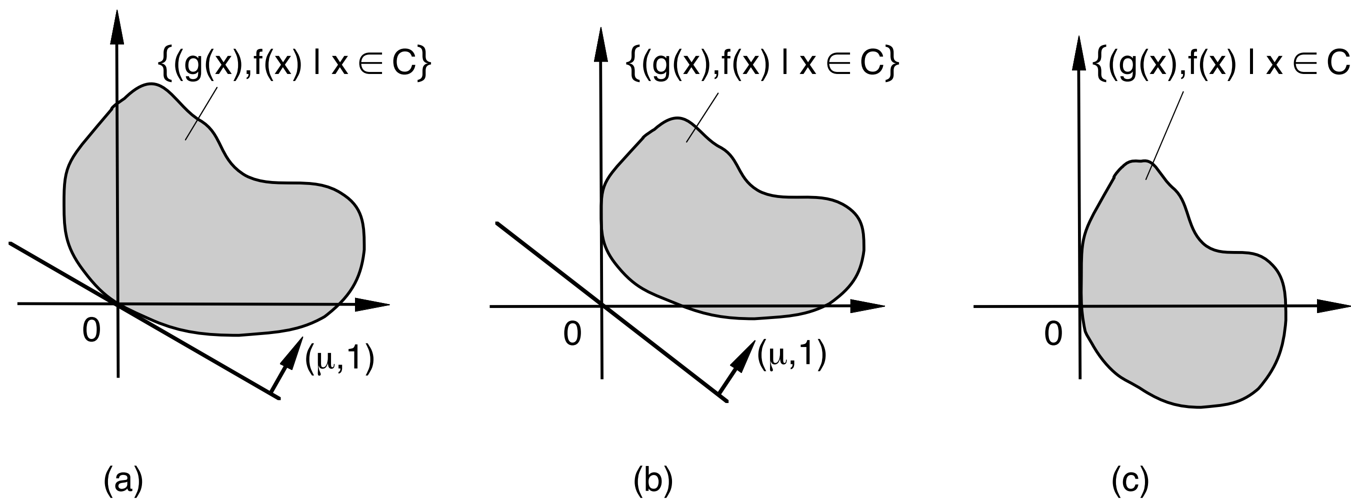
(2) The functions g_j, $j = 1, \ldots, r$, are affine, and F contains a relative interior point of C.

Then, there exist scalars $\mu_j^* \geq 0$, $j = 1, \ldots, r$, s. t.

$$f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \geq 0, \qquad \forall\, x \in C.$$

- Reduces to Farkas' Lemma if $C = \Re^n$, and f and g_j are linear.

VISUALIZATION OF NONLINEAR FARKAS' LEMMA



- Assuming that for all $x \in C$ with $g(x) \leq 0$, we have $f(x) \geq 0$, etc

- The lemma asserts the existence of a nonvertical hyperplane in $\Re^{r+1}$, with normal $(\mu, 1)$, that passes through the origin and contains the set

$$\big\{ \big(g(x), f(x)\big) \mid x \in C \big\}$$

in its positive halfspace.

- Figures (a) and (b) show examples where such a hyperplane exists, and figure (c) shows an example where it does not.

PROOF OF NONLINEAR FARKAS' LEMMA

- Apply Min Common/Max Crossing to

$$M = \big\{(u, w) \mid \text{there is } x \in C \text{ s. t. } g(x) \le u,\ f(x) \le w\big\}$$

- Under condition (1), Min Common/Max Crossing Theorem II applies: $0 \in \mathrm{ri}(D)$, where

$$D = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \overline{M}\big\}.$$

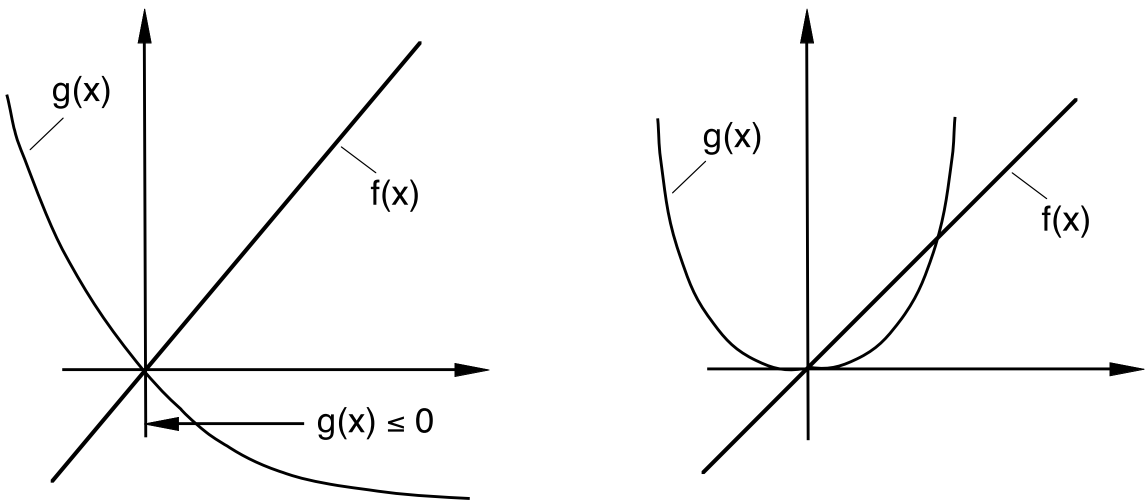
- Under condition (2), Min Common/Max Crossing Theorem III applies: $g(x) \le 0$ can be written as $Ax - b \le 0$.

- Hence for some μ^*, we have $w^* = \sup_\mu q(\mu) = q(\mu^*)$, where $q(\mu) = \inf_{(u,w)\in M}\{w + \mu' u\}$. Using the definition of M,

$$q(\mu) = \begin{cases} \inf_{x\in C}\big\{f(x) + \sum_{j=1}^r \mu_j g_j(x)\big\} & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise,} \end{cases}$$

so $\mu^* \ge 0$ and $\inf_{x\in C}\Big\{f(x) + \sum_{j=1}^r \mu_j^* g_j(x)\Big\} = w^* \ge 0$.

EXAMPLE



- Here $C = \Re$, $\qquad f(x) = x$. In the example on the left, g is given by $g(x) = e^{-x} - 1$, while in the example on the right, g is given by $g(x) = x^2$.

- In both examples, $f(x) \geq 0$ for all x such that $g(x) \leq 0$.

- On the left, condition (1) of the Nonlinear Farkas Lemma is satisfied, and for $\mu^* = 1$, we have

$$f(x) + \mu^* g(x) = x + e^{-x} - 1 \geq 0, \qquad \forall\, x \in \Re$$

- On the right, condition (1) is violated, and for every $\mu^* \geq 0$, the function $f(x) + \mu^* g(x) = x + \mu^* x^2$ takes negative values for x negative and sufficiently close to 0.

APPLICATION TO CONVEX PROGRAMMING

Consider the problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x \in F = \big\{x \in C \mid g_j(x) \le 0,\ j = 1,\dots,r\big\}, \end{aligned}$$

where $C \subset \Re^n$ is convex, and $f : C \mapsto \Re$ and $g_j : C \mapsto \Re$ are convex. Assume that f^* is finite.

- Replace $f(x)$ by $f(x) - f^*$ and apply the nonlinear Farkas Lemma. Then, under the assumptions of the lemma, there exist $\mu_j^* \ge 0$, such that

$$f^* \le f(x) + \sum_{j=1}^r \mu_j^* g_j(x), \qquad \forall\, x \in C.$$

Since $F \subset C$ and $\mu_j^* g_j(x) \le 0$ for all $x \in F$,

$$f^* \le \inf_{x \in F} \left\{ f(x) + \sum_{j=1}^r \mu_j^* g_j(x) \right\} \le \inf_{x \in F} f(x) = f^*.$$

Thus equality holds throughout, and we have

$$f^* = \inf_{x \in C} \left\{ f(x) + \sum_{j=1}^r \mu_j^* g_j(x) \right\}.$$

CONVEX PROGRAMMING DUALITY - OUTLINE

- Define the dual function

$$q(\mu) = \inf_{x \in C} \left\{ f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \right\}$$

and the dual problem $\max_{\mu \geq 0}\ q(\mu)$.

- Note that for all $\mu \geq 0$ and $x \in C$ with $g(x) \leq 0$

$$q(\mu) \leq f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \leq f(x)$$

Therefore, we have the $\mathit{weak\ duality}$ relation

$$q^* = \sup_{\mu \geq 0} q(\mu) \leq \inf_{x \in C,\, g(x) \leq 0} f(x) = f^*.$$

- If we can use Farkas' Lemma, there exists $\mu^* \geq 0$ that solves the dual problem and $q^* = f^*$.

- This is so if (1) there exists $\overline{x} \in C$ with $g_j(\overline{x}) < 0$ for all j, or (2) the constraint functions g_j are affine and there is a feasible point in $\mathrm{ri}(C)$.