LECTURE 10

LECTURE OUTLINE

- Polar cones and polar cone theorem
- Polyhedral and finitely generated cones
- Farkas Lemma, Minkowski-Weyl Theorem
- Polyhedral sets and functions

- The main convexity concepts so far have been:
  - Closure, convex hull, affine hull, relative interior, directions of recession
  - Preservation of closure under linear transformation and partial minimization
  - Existence of optimal solutions
  - Hyperplanes
  - Min common/max crossing duality and application in minimax

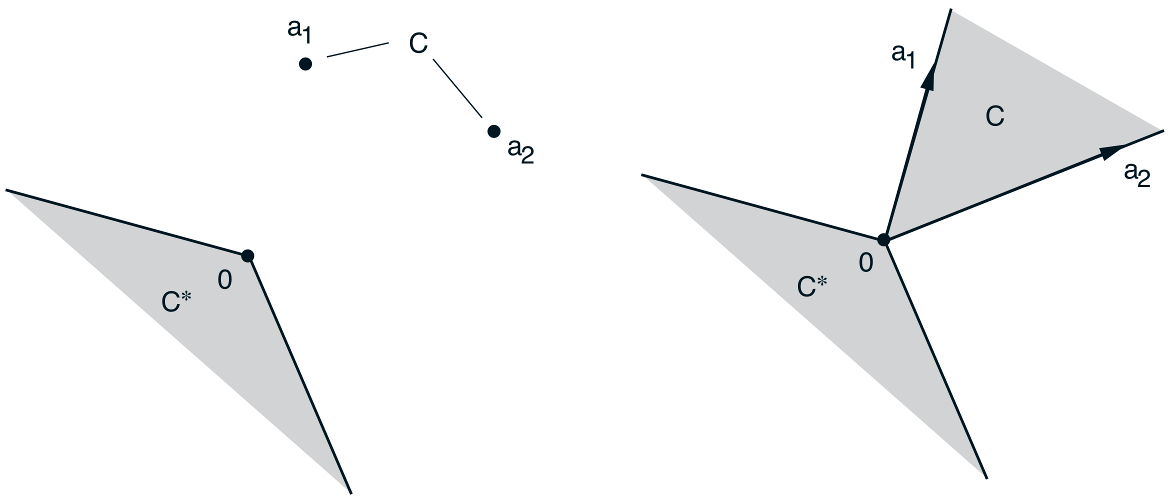
- We now introduce a new concept with important theoretical and algorithmic implications: polyhedral convexity, extreme points, and related issues.

POLAR CONES

- Given a set C, the cone given by

$$C^* = \{y \mid y'x \leq 0,\ \forall\, x \in C\},$$

is called the *polar cone* of C.



(a) (b)

- C^* is a closed convex cone, since it is the intersection of closed halfspaces.

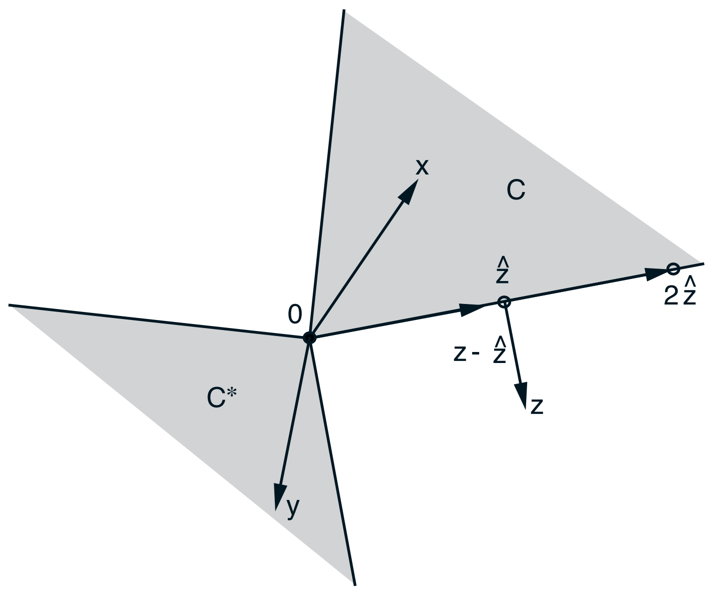
- Note that

$$C^* = \big(\mathrm{cl}(C)\big)^* = \big(\mathrm{conv}(C)\big)^* = \big(\mathrm{cone}(C)\big)^*.$$

- Important example: If C is a subspace, $C^* = C^\perp$. In this case, we have $(C^*)^* = (C^\perp)^\perp = C$.

POLAR CONE THEOREM

- For any cone C, we have $(C^*)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big)$. If C is closed and convex, we have $(C^*)^* = C$.



Proof: Consider the case where C is closed and convex. For any $x \in C$, we have $x'y \leq 0$ for all $y \in C^*$, so that $x \in (C^*)^*$, and $C \subset (C^*)^*$.

To prove the reverse inclusion, take $z \in (C^*)^*$, and let $\hat{z}$ be the projection of z on C, so that $(z - \hat{z})'(x - \hat{z}) \leq 0$, for all $x \in C$. Taking $x = 0$ and $x = 2\hat{z}$, we obtain $(z - \hat{z})'\hat{z} = 0$, so that $(z - \hat{z})'x \leq 0$ for all $x \in C$. Therefore, $(z - \hat{z}) \in C^*$, and since $z \in (C^*)^*$, we have $(z - \hat{z})'z \leq 0$. Subtracting $(z - \hat{z})'\hat{z} = 0$ yields $\|z - \hat{z}\|^2 \leq 0$. It follows that $z = \hat{z}$ and $z \in C$, implying that $(C^*)^* \subset C$.

POLYHEDRAL AND FINITELY GENERATED CONES

- A cone $C \subset \Re^n$ is *polyhedral*, if

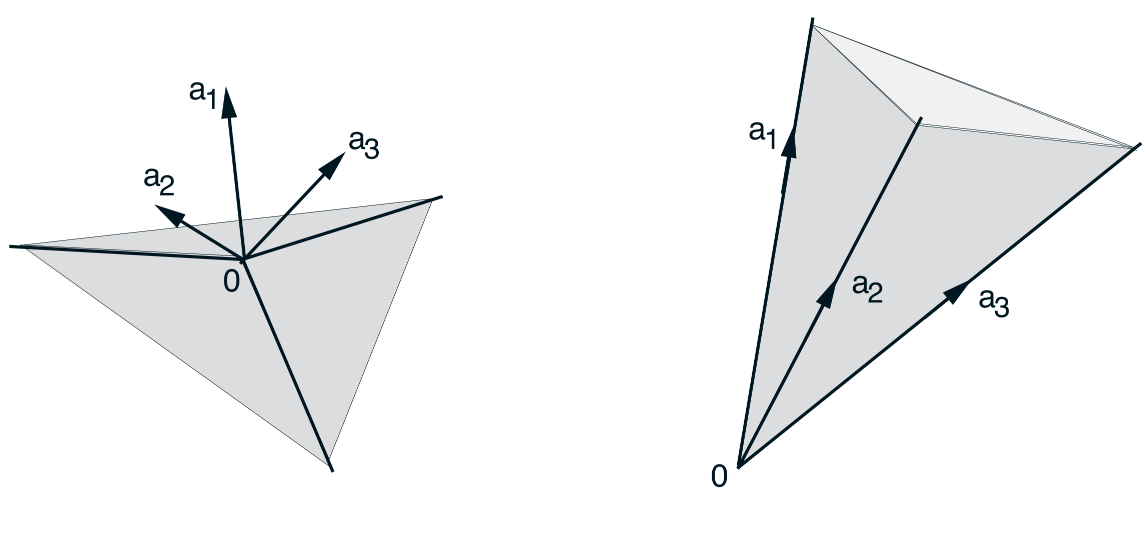
$$C = \{x \mid a_j'x \le 0,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$.

- A cone $C \subset \Re^n$ is *finitely generated*, if

$$C = \left\{ x \,\middle|\, x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \ge 0,\ j = 1, \ldots, r \right\}$$
$$= \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big),$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$.



(a) (b)

FARKAS-MINKOWSKI-WEYL THEOREMS

Let $a_1, \ldots, a_r$ be given vectors in $\Re^n$, and let

$$C = \text{cone}\big(\{a_1, \ldots, a_r\}\big).$$

(a) C is closed and

$$C^* = \big\{y \mid a_j'y \leq 0,\ j = 1, \ldots, r\big\}.$$

(b) (*Farkas' Lemma*) We have

$$\big\{y \mid a_j'y \leq 0,\ j = 1, \ldots, r\big\}^* = C.$$

(There is also a version of this involving sets described by linear equality as well as inequality constraints.)

(c) (*Minkowski-Weyl Theorem*) A cone is polyhedral if and only if it is finitely generated.

PROOF OUTLINE

(a) First show that for $C = \text{cone}(\{a_1, \ldots, a_r\})$,

$$C^* = \text{cone}(\{a_1, \ldots, a_r\})^* = \{y \mid a_j' y \leq 0,\ j = 1, \ldots, r\}$$

If $y' a_j \leq 0$ for all j, then $y' x \leq 0$ for all $x \in C$, so $C^* \supset \{y \mid a_j' y \leq 0,\ j = 1, \ldots, r\}$. Conversely, if $y \in C^*$, i.e., if $y' x \leq 0$ for all $x \in C$, then, since $a_j \in C$, we have $y' a_j \leq 0$, for all j. Thus, $C^* \subset \{y \mid a_j' y \leq 0,\ j = 1, \ldots, r\}$.

- Showing that $C = \text{cone}(\{a_1, \ldots, a_r\})$ is closed is nontrivial! Follows from Prop. 1.5.8(b), which shows (as a special case where $C = \Re^n$) that closedness of polyhedral sets is preserved by linear transformations. (The text has two other lines of proof.)

(b) Assume no equalities. Farkas' Lemma says:

$$\{y \mid a_j' y \leq 0,\ j = 1, \ldots, r\}^* = C$$

Since by part (a), $C^* = \{y \mid a_j' y \leq 0,\ j = 1, \ldots, r\}$ and C is closed and convex, the result follows by the Polar Cone Theorem.

(c) See the text.

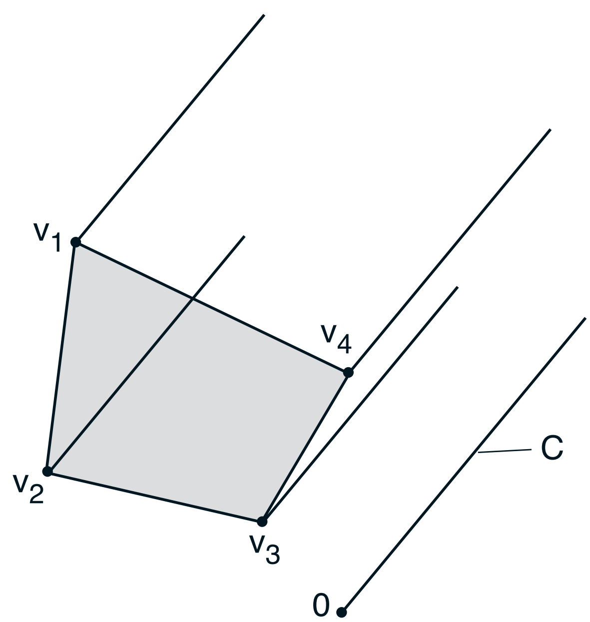
POLYHEDRAL SETS

- A set $P \subset \Re^n$ is said to be *polyhedral* if it is nonempty and

$$P = \{x \mid a_j' x \le b_j,\ j = 1, \ldots, r\},$$

for some $a_j \in \Re^n$ and $b_j \in \Re$.

- A polyhedral set may involve affine equalities (convert each into two affine inequalities).



Theorem: A set P is polyhedral if and only if

$$P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + C,$$

for a nonempty finite set of vectors $\{v_1, \ldots, v_m\}$ and a finitely generated cone C.

PROOF OUTLINE

Proof: Assume that P is polyhedral. Then,

$$P = \{x \mid a_j' x \le b_j,\ j = 1, \ldots, r\},$$

for some a_j and b_j. Consider the polyhedral cone

$$\hat{P} = \{(x, w) \mid 0 \le w,\ a_j' x \le b_j w,\ j = 1, \ldots, r\}$$

and note that $P = \{x \mid (x, 1) \in \hat{P}\}$. By Minkowski-Weyl, $\hat{P}$ is finitely generated, so it has the form

$$\hat{P} = \left\{ (x, w) \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j,\ w = \sum_{j=1}^{m} \mu_j d_j,\ \mu_j \ge 0 \right\},$$

for some v_j and d_j. Since $w \ge 0$ for all vectors $(x, w) \in \hat{P}$, we see that $d_j \ge 0$ for all j. Let

$$J^+ = \{j \mid d_j > 0\}, \qquad J^0 = \{j \mid d_j = 0\}.$$

PROOF CONTINUED

- By replacing μ_j by μ_j/d_j for all $j \in J^+$,

$$\hat{P} = \left\{ (x, w) \;\middle|\; x = \sum_{j \in J^+ \cup J^0} \mu_j v_j,\ w = \sum_{j \in J^+} \mu_j,\ \mu_j \geq 0 \right\}$$

Since $P = \{x \mid (x, 1) \in \hat{P}\}$, we obtain

$$P = \left\{ x \;\middle|\; x = \sum_{j \in J^+ \cup J^0} \mu_j v_j,\ \sum_{j \in J^+} \mu_j = 1,\ \mu_j \geq 0 \right\}$$

Thus,

$$P = \operatorname{conv}\big(\{v_j \mid j \in J^+\}\big) + \left\{ \sum_{j \in J^0} \mu_j v_j \;\middle|\; \mu_j \geq 0,\ j \in J^0 \right\}.$$

- To prove that the vector sum of $\operatorname{conv}\big(\{v_1, \ldots, v_m\}\big)$ and a finitely generated cone is a polyhedral set, we reverse the preceding argument. **Q.E.D.**

POLYHEDRAL FUNCTIONS

- A function $f : \Re^n \mapsto (-\infty, \infty]$ is *polyhedral* if its epigraph is a polyhedral set in $\Re^{n+1}$.
- Note that every polyhedral function is closed, proper, and convex.

Theorem: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a convex function. Then f is polyhedral if and only if $\mathrm{dom}(f)$ is a polyhedral set, and

$$f(x) = \max_{j=1,\ldots,m} \{a_j' x + b_j\}, \qquad \forall\, x \in \mathrm{dom}(f),$$

for some $a_j \in \Re^n$ and $b_j \in \Re$.

Proof: Assume that $\mathrm{dom}(f)$ is polyhedral and f has the above representation. We will show that f is polyhedral. The epigraph of f can be written as

$$\begin{aligned} \mathrm{epi}(f) = &\big\{(x, w) \mid x \in \mathrm{dom}(f)\big\} \\ &\cap \big\{(x, w) \mid a_j' x + b_j \le w,\ j = 1, \ldots, m\big\}. \end{aligned}$$

Since the two sets on the right are polyhedral, $\mathrm{epi}(f)$ is also polyhedral. Hence f is polyhedral.

PROOF CONTINUED

- Conversely, if f is polyhedral, its epigraph is a polyhedral and can be represented as the intersection of a finite collection of closed halfspaces of the form $\{(x,w) \mid a_j'x + b_j \le c_j w\}$, $j = 1,\ldots,r$, where $a_j \in \Re^n$, and $b_j, c_j \in \Re$.

- Since for any $(x,w) \in \text{epi}(f)$, we have $(x, w + \gamma) \in \text{epi}(f)$ for all $\gamma \ge 0$, it follows that $c_j \ge 0$, so by normalizing if necessary, we may assume without loss of generality that either $c_j = 0$ or $c_j = 1$. Letting $c_j = 1$ for $j = 1,\ldots,m$, and $c_j = 0$ for $j = m+1,\ldots,r$, where m is some integer,

$$\text{epi}(f) = \big\{(x,w) \mid a_j'x + b_j \le w,\ j = 1,\ldots,m,\\ a_j'x + b_j \le 0,\ j = m+1,\ldots,r\big\}.$$

Thus

$$\text{dom}(f) = \big\{x \mid a_j'x + b_j \le 0,\ j = m+1,\ldots,r\big\},$$

$$f(x) = \max_{j=1,\ldots,m}\{a_j'x + b_j\}, \qquad \forall\, x \in \text{dom}(f).$$

Q.E.D.