MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2005 6.436J/15.085J

Midterm exam (120 mins/100 pts) 10/25/05

All parts can be done independently

Problem 1: (10 points)
For the questions below, provide a brief justification for your answer.

(a) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of σ-fields of subsets of a set Ω, with the property $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. Is $\cup_{n=1}^{\infty} \mathcal{F}_n$ a σ-field?

(b) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of σ-fields of subsets of a set Ω. Is $\cap_{n=1}^{\infty} \mathcal{F}_n$ a σ-field?

Solution:

(a) $\cup_{n=1}^{\infty} \mathcal{F}_n$ need not be a σ-field. The infinite coin toss model provides an example.

(b) $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_n$ is a σ-field. There are three properties to check:

- $\emptyset$ and Ω are in all $\mathcal{F}_n$. Therefore, they are in their intersection.
- Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_n$ for all n. Consequently, $A^c \in \mathcal{F}_n$ for all n. Therefore, $A^c \in \mathcal{F}$.
- Let $(A_i)_i$ be a countable family of sets in $\mathcal{F}$. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_n$ for all n. Therefore, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Problem 2: (10 points)
A probabilistic experiment involves an infinite sequence of trials. For $k = 1, 2, \ldots$, let A_k be the event that the kth trial was a success. Write down a set-theoretic expression that describes the following event:

B : For every k there exists an ℓ such that trials $k\ell$ and $k\ell^2$ were both successes.

Note: A "set theoretic expression" is an expression like $\cup_{k>5} \cap_{\ell<k} A_{k+\ell}$.
Solution:
$B = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{\infty} (A_{kl} \cap A_{kl^2})$.

Problem 3: (10 points)
Alice has a matchbox in each pocket. Initially, each box contains n matches. At each time step, she reaches into the left pocket (with probability p) or into the right pocket (with probability $1-p$), and removes a match. Her choices at different times are independent. Let K be the first time that the match removed happens to be the last match of the selected box. Find the probability mass function of K.

Solution:

Only two cases can happen the last match is picked in the right pocket (event R) or in the left pocket (event L).

Observe that $n \leq K \leq 2n-1$. Let $k \in \{0, \ldots, n-1\}$. If $K = n+k$, k picks in the non-emptied pocket have been chosen in the first $n+k-1$ picks, since the last pick was in the emptied pocket.

$$\mathbf{P}(K = n+k,\ L) = \binom{n+k-1}{k} p^n (1-p)^k$$
$$\mathbf{P}(K = n+k,\ R) = \binom{n+k-1}{k} (1-p)^n p^k$$

Consequently, $\mathbf{P}(K = n+k) = \binom{n+k-1}{k}(p^n(1-p)^k + (1-p)^n p^k)$.

Problem 4: (15 points)
A 4-sided die has its four faces labeled as a, b, c, d. Each time the die is rolled, the result is a, b, c, or d, with probabilities p_a, p_b, p_c, p_d, respectively. Different rolls are statistically independent. The die is rolled n times. Let N_a and N_b be the number of rolls that resulted in a or b, respectively. Find the covariance of N_a and N_b. *Hint:* Use indicator variables.
Solution:

Let $\mathbf{1}_l^i,\ l = a, b, c, d$ be the indicator function for the ith roll. With this definition, $N_l = \sum_{i=1}^n \mathbf{1}_l^i$ and $N_l = np_l$.

$$\begin{aligned} \mathbf{E}[N_a N_b] &= \mathbf{E}\left[\sum_{i=1}^n \mathbf{1}_a^i \mathbf{1}_b^i + 2\sum_{i<j} \mathbf{1}_a^i \mathbf{1}_b^j\right] \\ &= 2\sum_{i<j} \mathbf{E}\left[\mathbf{1}_a^i \mathbf{1}_b^j\right] \\ &= 2\sum_{i<j} p_a p_b \\ &= n(n-1)p_a p_b. \end{aligned}$$

$$\mathrm{Cov}(N_a, N_b) = \mathbf{E}[N_a N_b] - \mathbf{E}[N_a]\mathbf{E}[N_b] = n(n-1)p_a p_b - n^2 p_a p_b = -np_a p_b$$

Problem 5: (10 points)
Let X and Y be independent exponential random variables with parameter $\lambda = 1$. That is, $f_X(t) = f_Y(t) = e^{-t}$, for $t \geq 0$. Let

$$U = X^2, \qquad V = X^2 + Y.$$

Find the joint PDF of U and V.
Solution:

X, Y have joint density $f_{X,Y}(x, y)$ equal to $\exp(-x-y)$ for $x, y \geq 0$, and zero otherwise.

The mapping $g : (X, Y) \to (U = X^2, V = X^2 + Y)$ is invertible from $[0, +\infty)^2$ into itself, and $X = \sqrt{U}$ and $Y = V - U$. The inverse mapping is also continuously differentiable when $u \neq 0$.

Using the change of variable formula, (U, V) has a density $f_{U,V}(u, v) = f_{X,Y}(\sqrt{u}, v-u)/2\sqrt{u}$ for $u, v > 0$ and zero otherwise.

Problem 6: (10 points)
Suppose that $\mathbf{E}[X \mid Y] = \mathbf{E}[X]$, with probability 1.

(a) Show that X and Y are uncorrelated.

(b) Give an example to show that X and Y need not be independent.

Solution:

(a) $\mathbf{E}[XY] = \mathbf{E}[\mathbf{E}[XY|Y]] = \mathbf{E}[Y\mathbf{E}[X|Y]] = \mathbf{E}[Y\mathbf{E}[X]] = \mathbf{E}[X]\mathbf{E}[Y]$. Consequently, X, Y are uncorrelated.

(b) Consider two discrete random variables. Y can take values -1 and 1, each with probability 1/2, while X takes values $-1, 1, 2$. Their joint distribution is given by the following table.

X	$Y = -1$	1
-1	1/2	2/3
1	1/2	0
2	0	1/3

It is easy to check that $E[X|Y = -1] = E[X|Y = 1] = 0$. However, $E[X^2|Y = -1] = 1$, whereas $E[X^2|Y = 1] = 2$. Hence, X, Y are not independent.

Problem 7: (15 points)
Let K be a discrete random variable taking values 1 and 2 with probability 1/2. Let X be a continuous random variable whose conditional distribution, given the event $K = k$, is normal with zero mean and variance k.

(a) Provide a formula with the PDF of X.

(b) Provide a formula for $\mathbf{P}(K = 1 \mid X = x)$.

Solution:

(a) $f_X(x) = \mathbf{P}(K = 1)f_{X|1}(x) + \mathbf{P}(K = 2)f_{X|2}(x) = \frac{1}{2\sqrt{2\pi}}\left(\exp(-x^2/2) + \frac{1}{\sqrt{2}}\exp(-x^2/4)\right)$.

(b) By Bayes rule, $\mathbf{P}(K = 1 \mid X = x) = \frac{f_{X|1}(x)\mathbf{P}(K=1)}{f_X(x)} = \frac{f_{X|1}(x)}{2f_X(x)}$.

Problem 8: (20 points)
Consider a sequence of events A_k, with the property that $\lim_{k\to\infty} \mathbf{P}(A_k) = 0$.

(a) Show that $\mathbf{P}$(all but finitely many of the A_k occur) $= 0$.

(b) Is it possible to have $\mathbf{P}$(infinitely many of the A_k occur) $= 1$?

(c) Suppose in addition that $\sum_{k=1}^{\infty} \mathbf{P}(A_k^c \cap A_{k+1}) < \infty$. Show that $\mathbf{P}$(infinitely many of the A_k occur) $= 0$.

Solution:

(a) $\mathbf{P}(\liminf A_k) = \lim_{n\to\infty} \mathbf{P}(\bigcap_{k\geq n} A_k)$ by continuity of probability. But $\mathbf{P}(\bigcap_{k\geq n} A_k) \leq \inf_{k\geq n} \mathbf{P}(A_k) \to 0$ as $n \to \infty$.

(b) $\mathbf{P}(\text{infinitely many of the } A_k \text{ occur}) = \mathbf{P}(\limsup A_k)$ could be one. This happens, for example, using the second Borel-Cantelli lemma, if the A_k are independent and $\mathbf{P}(A_k) = 1/k$.

(c) Define the set

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

We wish to show $\mathbf{P}(A) = 0$. Now, $A \subseteq \cup_{m=n}^{\infty} A_m$ for all m, and by monotonicity of the measure, $\mathbf{P}(A) \leq \mathbf{P}(\cup_{m=n}^{\infty} A_m)$, for all n. In addition,

$$\begin{aligned}
\bigcup_{m=n}^{\infty} A_m &= A_n \cup (A_{n+1} \setminus A_n) \cup (A_{n+2} \setminus A_{n+1}) \cup \cdots \\
&= A_n \cup (A_{n+1} \cap A_n^c) \cup (A_{n+2} \cap A_{n+1}^c) \cup \cdots,
\end{aligned}$$

Therefore by the union bound, for all n,

$$\begin{aligned}
\mathbf{P}(A) &\leq \mathbf{P}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&\leq \mathbf{P}(A_n) + \sum_{m=n}^{\infty} \mathbf{P}(A_{m+1} \cap A_m^c).
\end{aligned}$$

This holds for all n, and therefore it holds in the limit as n goes to infinity. But the limit of the final expression is zero, since $\mathbf{P}(A_n) \to 0$, and since $\sum_{n=1}^{\infty} \mathbf{P}(A_n^c \cap A_{n+1}) < \infty$.