6.441 Transmission of Information
Final Exam

Spring 2003

Problem 1 (20pt) For each of the statements below, decide if it is TRUE or FALSE. Give a proof if true, and a counter example or prove an alternative statement if false.

a)(4pt) U and V are independent random variables. Let $X = f(U)$ for an arbitrary function, and $Y = f(U) + V$, then

$$I(X;Y) = I(U;Y)$$

b)(4pt) U and V are i.i.d. binary random variables, taking value in $\{0,1\}$, $W = \max\{U,V\}$, then

$$H(U,V) \geq H(W,V)$$

c)(4pt) For part **b)**,

$$H(U) \geq H(W)$$

d)(4pt) The "mutual information between three random variables", X, Y, and Z is defined as $I(X;Y;Z) = I(X;Y) - I(X;Y|Z)$.

This quantity is symmetric in the three random variables, i.e.,

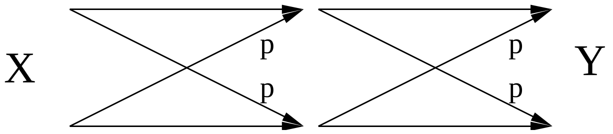
$$I(X;Y;Z) = I(X;Z) - I(X;Z|Y)$$

e)(4pt) In part **d)**, for any X, Y, Z,

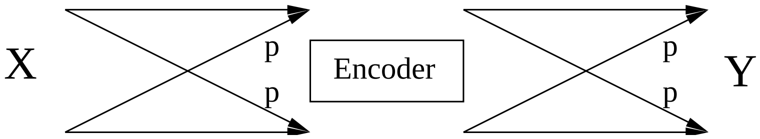
$$I(X;Y;Z) \geq 0$$

Problem 2 (15 pt) Compute the capacity of the following channels

a)(4 pt) Concatenation of 2 BSC's $Y = X \oplus Z_1 \oplus Z_2$, where $X \in \{0,1\}$, and Z_1, Z_2 are i.i.d. Bernoulli(p)



b)(4 pt) Concatenations of BSC's with encoding and decoding between each other



c)(7 pt) 2 independent looks at X: $\underline{Y} = (X \oplus Z_1, X \oplus Z_2)$, again Z_i's are i.i.d Bern(p).

Problem 3 (15 pt) Consider a wireless channel with K transmit antennas and one receive antenna. The channel gain between the i^{th} transmit antenna and the receive antenna is H_i. The encoder maps the incoming message $M \in \{1, 2, \ldots, 2^{nR}\}$ into a K dimensional vector $\underline{X} = [X_1, X_2, \ldots, X_K]^T$ and transmit over the antennas. The total transmitted power constraint is P, i.e.,

$$\sum_i E[|X_i|^2] \leq P$$

The receiver receives a scalar symbol

$$Y = \sum_{i=1}^{K} H_i X_i + W$$

where $W \sim N(0, \sigma^2)$ is the additive Gaussian noise. H_i's are fixed constants. Notice the channel is memoryless, and the time index is omitted.

a)(6 pt) Assume that both the transmitter and the receiver know the channel parameters H_i's. Find the optimal input distribution and compute the channel capacity.

b)(5 pt) Assume now that the receiver knows H_i's perfectly, but the transmitter does not. It makes sense to choose X_i's to have zero-mean i.i.d. Gaussian distribution (each entry has variance P/K). Compute the achievable data rate using this input strategy.

c)(4 pt) Compare the performance in **a)** and **b)**, write an expression for the percentage loss in capacity from not knowing the channel at the transmitter. Make appropriate approximations at high and low SNR. In which regime is the loss greater?

Problem 4 (30 pt) Consider a parallel Gaussian multiple access channel as follows,

$$Y_i = a_i U_i + b_i V_i + W_i, i = 1, \ldots, k$$

where $a_i, b_i, i = 1, \ldots, k$ are fixed constants, known to both the transmitter and the receiver. W_i's are i.i.d. Gaussian noise with variance σ^2. $\underline{U}, \underline{V}$ are the symbols transmitted by two users, denoted as user A and B, respectively. Again, the time index is omitted for this memoryless channel. Let the power constrains for the individual users be $E[\|\underline{U}\|^2] \leq P$, and $E[\|\underline{V}\|^2] \leq Q$.

As in the scalar multiple access channels, the capacity region C is defined as the closure of the convex hull of the rate pairs (R_A, R_B) that satisfies.

$$\begin{aligned} R_A &\leq I(\underline{U}; \underline{Y}|\underline{V}) \\ R_B &\leq I(\underline{V}; \underline{Y}|\underline{U}) \\ R_A + R_B &\leq I(\underline{U}, \underline{V}; \underline{Y}) \end{aligned}$$

for some input distribution $p_{\underline{U}}, p_{\underline{V}}$ satisfying the power constraints.

a) (7 pt) Compute the the maximum achievable data rate for user A,

$$C_A = \max_{(R_A, R_B) \in C} R_A$$

Find an input distribution that achieves this maximum. Is it necessary that user B keep silent to achieve this maximum?

b)(8 pt) Show that to compute the capacity region, it suffices to consider only the input distributions that $\underline{U}, \underline{V}$ are both Gaussian random vectors with independent entries (covariance matrices are diagonal). i.e., any rate pair (R_A, R_B) that is achievable with any input distributions can be achieved with independent Gaussian distributions.

c) (5 pt) Compute the maximum achievable sum rate

$$C_{sum} = \max_{(R_A, R_B) \in C} R_A + R_B$$

and find the optimal input distribution

Hint: The optimal distribution (to maximize the sum rate) has only one user transmit in each of the parallel channels. You need to find an optimal input in this form, and verify that it is indeed optimal using the Lagrange multiplier method.

d) (5 pt) Is there a "global" optimal input distribution? That is, is there a particular input distribution that achieves all rate pairs in the capacity region? Explain.

e)(5 pt) Now assume $k = 2$, $a_1 = b_2 = \sqrt{2}, a_2 = b_1 = 1$, $\sigma^2 = 0$. $P = 10$, $Q = 2$. Assume now user A uses a distribution that U_1, U_2 are i.i.d. Gaussian with variance $P_1 = P_2 = P/2 = 5$. All these parameters are known to both users and the receiver. Assume that $\underline{V}$ has to be independent of $\underline{U}$, and user B tries to jam the transmission of user A (to minimize the mutual information), what strategy should user B use? Explain how your answer will change if user B knows precisely the data and the signal that user A transmits.

Problem 5 (20 pt) Consider two memoryless channels $(\mathcal{X}_1, \mathcal{Y}_1, p_1(y_1|x_1))$ and $(\mathcal{X}_2, \mathcal{Y}_2, p_1(y_2|x_2))$. The two channels are independent, i.e.,

$$P(y_1, y_2|x_1, x_2) = p_1(y_1|x_1)p_2(y_2|x_2)$$

Let the capacity of the two channels be C_1, C_2 respectively.

Consider a coding and decoding scheme as follows,

Let the message be

$$M \in \{1, 2, \ldots, 2^{nR}\}$$

Encoder:

1. Randomly choose a codebook $\mathcal{C}_1$ of 2^{nR} codewords in $\mathcal{X}_1^n$, each entry of each codeword is chosen i.i.d. according to the capacity-achieving distribution of channel 1.
2. Randomly assign these codewords into $2^{nR'}$ bins, each codeword is assigned uniformly and independently to a bin.
3. Randomly choose a codebook $\mathcal{C}_2$ of $2^{nR'}$ codewords in $\mathcal{X}_2^n$, each entry of each codeword is chosen i.i.d. according to the capacity achieving distribution of channel 2.
4. Reveal the codebooks and the binning to the decoder.
5. To transmit a message M, find the corresponding codeword in $\mathcal{C}_1$ to transmit over channel 1, and transmit the bin number over channel 2

Decoder receives $\underline{y}_1, \underline{y}_2$. From $\underline{y}_2$, the sequence of $\underline{x}_2$ is decoded to find the bin number. Then find a codeword $\tilde{\underline{x}}_1$ in the corresponding bin that is jointly typical with $\underline{y}_1$. Decode if there is unique such codeword, and declare error otherwise.

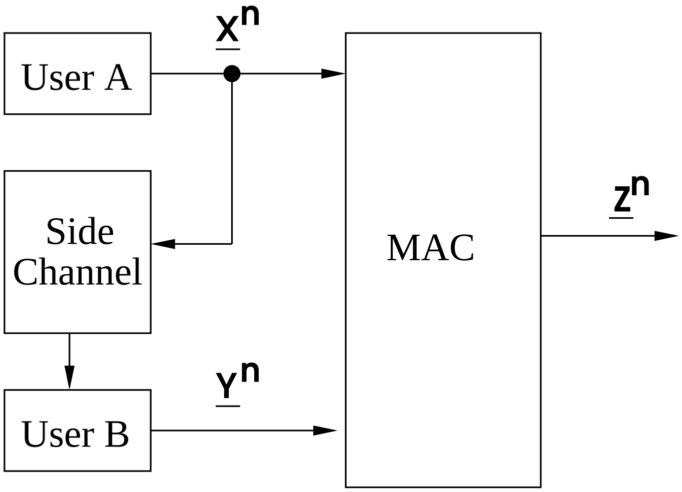
a)(10 pt) Show that this strategy is optimal, i.e., as long as $R < C_1 + C_2$, one can find an appropriate R', for which the decoding error probability of the above scheme goes to 0 as n increases.

b)(10 pt) Now consider a multiple access channel with input X, Y and output Z as shown in the figure

Assume that (R_A, R_B) are a pair of achievable data rates for the MAC with particular input distributions $p_A(x), p_B(y)$.

Now user B listens to the signal $\underline{X}^n$ transmitted by user A through a side channel, and tries to help user A to transmit a higher data rate. The mutual information of the side channel with input distribution p_A is larger than $R_A + R_B$. Design a scheme using part **a)** such that user A can reliably transmit at a rate $R = R_A + R_B$. Argue that the collaboration between the users enlarges the capacity region. Intuitively explain what information should be passed through the side channel.

(Hint, you can ignore the fact that the side channel is causal, and allows user B to "listen" to user A's signal ahead of time. You will get a bonus if you can design a scheme that can achieve $R = R_A + R_B$ with a causal side channel. The key is to divide the time axis into multiple long blocks.)



User A
$\underline{X}^n$
Side Channel
User B
$\underline{Y}^n$
MAC
$\underline{Z}^n$