6.441 Transmission of Information
Problem Set 6

Spring 2003

Due date: April 10

Problem 1, Coordinate Systems Let X_1, X_2 be i.i.d. $N(0, \sigma^2)$ distributed. Let $R = \sqrt{X_1^2 + X_2^2}$ and $T = R^2$.

a) Use Lagrange method to show that the distribution that maximizes differential entropy subject to the mean value constraint

$$\max_{f_X : E[X] \leq \mu, X \geq 0} h(X)$$

is the exponential distribution.

b) Show that R has Rayleigh distribution, i.e.

$$f_R(r) = \frac{r}{\sigma^2} \exp(-\frac{r^2}{2\sigma^2})$$

c) Show that T has exponential distribution.

d) Use a coordinate change to show that the distribution that solves the following optimization problem is the Rayleigh distribution.

$$\max_{f_R : R \geq 0, E[R^2] \leq 2\sigma^2} h(R) + E[\log R]$$

e) Use part b), c), and d) to prove part a) in a different way.

Problem 2 Suppose X and Y are real-valued, independent, random variables, use Jensen's inequality to prove that

$$h(X + Y) \geq h(X)$$

Problem 3 Use the fact that a Gaussian vector $\underline{X}$ with zero mean and covariance matrix K has $h(\underline{X}) = \frac{1}{2} \log(2\pi e)^n \det(K)$ to show that for any positive semi-definite symmetric matrices K,

a) $\det(K) \leq \prod_i K_{ii}$, where K_{ii} is the i^{th} diagonal element of K, with equality iff K is diagonal.

b) For any $\lambda \in [0,1]$,

$$\det(\lambda K_1 + (1-\lambda)K_2) \geq \det(K_1)^{\lambda} \det(K_2)^{1-\lambda}$$

c) $\det(K_1 + K_2) \geq \det(K_1)$.

Problem 4 Problem 10.2 in Cover and Thomas

Problem 5 Let $\underline{X}_1, \underline{X}_2$ be two Gaussian random vectors of dimension n. Let their means be $\underline{\mu}_1, \underline{\mu}_2$, and covariances be K_1 and K_2, respectively. Show that

$$D(\underline{X}_1 || \underline{X}_2) = \frac{1}{2}\text{trace } K_2^{-1}[K_1 - K_2 + (\underline{\mu}_1 - \underline{\mu}_2)(\underline{\mu}_1 - \underline{\mu}_2)^T] + \frac{1}{2}\log\frac{\det(K_2)}{\det(K_1)}$$