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6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

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6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Quiz 1 - Solutions 2003	
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Problem 1

A

Question: What is the potential distribution, $\Phi(r=0,z)$, along the z axis for $z > b$?

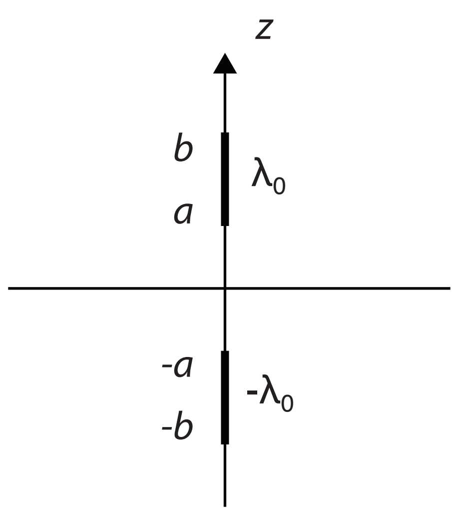

Figure 1: Line charge λ_0 above $z=0$ ground plane and image line charge $-\lambda_0$ (Image by MIT OpenCourseWare.)

Image line charge $-\lambda_0$ extends from $-a < z < -b$. Take charge elements $\pm\lambda_0 dz'$ at $z=z'$ and $z=-z'$. For $z > b$, $z > z'$.

Solution:

$$
\begin{aligned}
d\Phi(r=0,z) &= \frac{\lambda_0}{4\pi\epsilon_0}\left[\frac{1}{z-z'} - \frac{1}{z+z'}\right]dz' \\
\Phi(r=0,z) &= \frac{\lambda_0}{4\pi\epsilon_0}\int_a^b \left[\frac{1}{z-z'} - \frac{1}{z+z'}\right]dz' \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\left[-\ln(z'-z) - \ln(z'+z)\right]\Big|_{z'=a}^{b} \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\left[-\ln(b-z) - \ln(b+z) + \ln(a-z) + \ln(a+z)\right] \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\ln\left[\frac{(a-z)(a+z)}{(b-z)(b+z)}\right] = \frac{\lambda_0}{4\pi\epsilon_0}\ln\left[\frac{a^2-z^2}{b^2-z^2}\right]
\end{aligned}
$$

B

Question: What is the electric field, $\bar{E}(r=0,z)$, along the z axis for $z>b$?

Solution:

$$\begin{aligned} E_z(r=0,z) = -\frac{\partial \Phi(r=0,z)}{\partial z} &= \frac{\lambda_0}{4\pi\epsilon_0}\left[-\frac{1}{b-z}+\frac{1}{b+z}+\frac{1}{a-z}-\frac{1}{a+z}\right] \\ &= \frac{\lambda_0 z}{2\pi\epsilon_0}\frac{(b^2-a^2)}{(a^2-z^2)(b^2-z^2)} \end{aligned}$$

C

Question: What is the free surface charge density, $\sigma_S(r, z=0)$, as a function of radial distance r on the $z=0$ ground plane? Hint: One or more of these integrals may be useful for solving this problem.

$$\int \frac{xdx}{\left[x^2+c^2\right]^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2+c^2}}; \int \frac{dx}{\left[x^2+c^2\right]^{\frac{3}{2}}} = -\frac{x}{c^2\sqrt{x^2+c^2}}$$

$$\int \frac{xdx}{\sqrt{x^2+c^2}} = \sqrt{x^2+c^2}; \int \frac{dx}{x\sqrt{x^2+c^2}} = -\frac{1}{c}\ln\left[\frac{c+\sqrt{x^2+c^2}}{x}\right]$$

Solution:

$\underline{z=0}$

$$dE_z = \frac{-\lambda_0 dz'}{4\pi\epsilon_0\left[z'^2+r^2\right]} 2\cos\theta; \cos\theta = \frac{z'}{\sqrt{z'^2+r^2}}$$

$$\begin{aligned} dE_z &= \frac{-\lambda_0 z' dz'}{2\pi\epsilon_0\left[z'^2+r^2\right]^{\frac{3}{2}}} \\ E_z &= \frac{-\lambda_0}{2\pi\epsilon_0}\int_a^b \frac{z'dz'}{\left[z'^2+r^2\right]^{\frac{3}{2}}} \\ &= \frac{\lambda_0}{2\pi\epsilon_0}\left.\frac{1}{\sqrt{z'^2+r^2}}\right|_{z'=a}^{b} \\ &= \frac{\lambda_0}{2\pi\epsilon_0}\left[\frac{1}{\sqrt{b^2+r^2}}-\frac{1}{\sqrt{a^2+r^2}}\right] \\ \sigma_s(z=0) = \epsilon_0 E_z(z=0_+) &= \frac{\lambda_0}{2\pi}\left[\frac{1}{\sqrt{b^2+r^2}}-\frac{1}{\sqrt{a^2+r^2}}\right] \end{aligned}$$

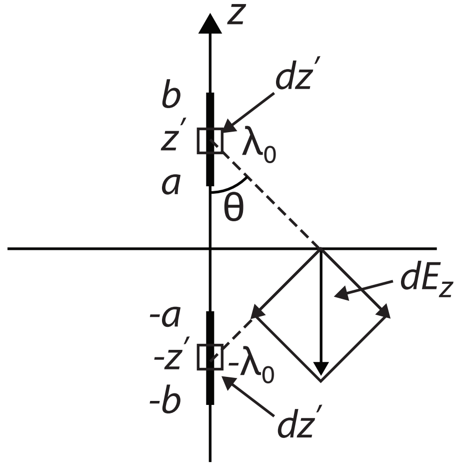

Figure 2: By symmetry each charge element and its image causes a $-z$ directed electric field. (Image by MIT OpenCourseWare.)

Problem 2

A

Question: What is the volume charge distribution for $0 < x < s$ as a function of time?

Solution:

$$\rho_f(t) = \rho_0 e^{-\frac{t}{\tau}}, \tau = \frac{\epsilon}{\sigma}$$

B

Question: What is the electric field $E_x(x,t)$?

Solution:

$$\nabla \cdot \bar{E} = \frac{\partial E_x}{\partial x} = \frac{\rho_f(t)}{\epsilon} \Rightarrow E_x = \frac{\rho_f(t)x}{\epsilon} + c(t)$$

$$\int_0^s E_x dx = \frac{\rho_f(t)s^2}{2\epsilon} + c(t)s = 0$$

$$c(t) = -\frac{\rho_f(t)s}{2\epsilon}$$

$$E_x = \frac{\rho_f(t)}{\epsilon}\left(x - \frac{s}{2}\right)$$

C

Question: What are the surface charge densities as a function of time at $x = 0$ and $x = s$?

Solution:

$$\sigma_s(x=0) = \epsilon E_x(x=0) = \frac{-\rho_f(t)s}{2}$$
$$\sigma_s(x=s) = -\epsilon E_x(x=s) = \frac{-\rho_f(t)s}{2}$$

D

Question: What is the current $i(t)$ flowing through the short circuit?

Solution:

$$\begin{aligned}\frac{i(t)}{A} &= \sigma E_x(x=s) + \epsilon \left.\frac{\partial E_x}{\partial t}\right|_{x=s} \\ &= \frac{\sigma \rho_f(t) s}{2\epsilon} + \frac{\not{\epsilon} s}{2\not{\epsilon}}\frac{\partial \rho_f}{\partial t} \\ &= \frac{\sigma s}{2\epsilon}\rho_0 e^{-\frac{t}{\tau}} + \frac{s}{2}\left(-\frac{1}{\tau}\right)\rho_0 e^{-\frac{t}{\tau}} \\ &= \frac{\sigma s}{2\epsilon}\rho_0 e^{-\frac{t}{\tau}} - \frac{s\sigma}{2\epsilon}\rho_0 e^{-\frac{t}{\tau}} \\ &= 0\end{aligned}$$

Problem 3

A

Question: Consider first the system shown in Figure 3.1 (see questions). In this system, a surface current exists only in the plane $z = 0$ and is everywhere surrounded by free space. For $x > 0$, the surface current is given by $\bar{K} = K_0\bar{i}_y$. For $x < 0$, the surface current is given by $\bar{K} = -K_0\bar{i}_y$. Which components of the magnetic field $\bar{H}$ do you expect to be non-zero in this system? On which coordinates (x and/or y and/or z) do you expect the non-zero components to depend? Explain briefly.

Solution: $\bar{J}$ is only in the $\hat{y}$ direction so therefore $\bar{A}$ is only in the $\hat{y}$ direction. The system is symmetric in y, so only x and z dependencies are expected. Therefore, $\bar{A} = A_y(x,y)\hat{y}$. Following $\bar{B} = \mu_0\bar{H} = \nabla \times \bar{A}$, $\bar{B} = \hat{z}\frac{\partial A_y}{\partial x} - \hat{x}\frac{\partial A_y}{\partial z}$. In summary, $\bar{B} = B_x(x,z)\hat{x} + B_z(x,z)\hat{z}$.

B

Question: Derive an integral expression for the magnetic field $\bar{H}$ but do not evaluate it.

Solution:

$$\bar{H}(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{K}(\bar{r}') \times (\bar{r} - \bar{r}')}{4\pi|\bar{r} - \bar{r}'|^3} dx' dy'$$
$$\bar{K}(\bar{r}') = \begin{cases} K_0\hat{y} & \text{for } x' > 0, \\ -K_0\hat{y} & \text{for } x' < 0 \end{cases}$$

$\bar{r} = x\hat{x} + z\hat{z}$ (y dependence is not necessary)
$\bar{r}' = x'\hat{x} + y'\hat{y}$ (integration only in the $x - y$ plane)

$$\bar{H}(\bar{r}) = \int_0^{\infty}\int_{-\infty}^{\infty} \frac{K_0\hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{\frac{3}{2}}} dy'dx' - \int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{K_0\hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{\frac{3}{2}}} dy'dx'$$

$$\bar{H}(x,z) = \int_0^{\infty}\int_{-\infty}^{\infty} \frac{K_0 z\hat{x} - K_0(x-x')\hat{z}}{2\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{\frac{3}{2}}} dy'dx'$$

C

Question: Consider now the system shown in Figure 3.2 (see questions). In this system, a surface current exists only in the plane $z = d$ above a perfect conductor that occupies the infinite half space $z < 0$. Free space occupies the infinite half-space $z > 0$. For $x > L$, the surface current is given by $\bar{K} = K_0\bar{i}_y$. For $x < -L$, the surface current is given by $\bar{K} = -K_0\bar{i}_y$. Writing the answer to Part (b) as $\bar{H}^*(x,y,z)$, which of the possible answers given below is the solution $\bar{H}(x,y,z)$ for the configuration shown in Figure 3.2?

i) $\bar{H}(x,y,z) = \left[\bar{H}^*(x-L,y,z-d) + \bar{H}^*(x+L,y,z-d) + \bar{H}^*(x-L,y,z+d) + \bar{H}^*(x+L,y,z+d)\right]$

ii) $\bar{H}(x,y,z) = \frac{1}{2}\left[\bar{H}^*(x-L,y,z-d) + \bar{H}^*(x+L,y,z-d) - \bar{H}(x-L,y,z+d) - \bar{H}^*(x+L,y,z+d)\right]$

iii) $\bar{H}(x,y,z) = \frac{1}{2}\left[\bar{H}^*(x-L,y,z-d) - \bar{H}^*(x+L,y,z+d)\right]$

iv) $\bar{H}(x,y,z) = \left[\bar{H}^*(x-L,y,z+d) + \bar{H}^*(x+L,y,z-d)\right]$

See questions for Figure 3.2

Solution: Let K^* be the distribution of the surface current in the $\hat{y}$ direction at $z = 0$ in Parts (A) and (B). Then, the distribution of the surface current in the $\hat{y}$ direction at $z = D$ in this part is given by $K = \frac{1}{2}\left(K^*(x-L,y) + K^*(x+L,y)\right)$. Here, superposition is used to create the new surface current distribution. The resulting magnetic fields are similarly superimposed, after translation by D in the $\hat{z}$ direction. A second set of superpositions is used to match the boundary condition at the perfect conductor following the method of images. Hence,

$$\bar{H} = \frac{1}{2}\left[\bar{H}^* \underbrace{(x-L,z-D) + \bar{H}^*(x+L,z-D)}_{\text{Translated and superimposed original solutions}} - \underbrace{\bar{H}^*(x-L,z+D) - \bar{H}^*(x+L,z+D)}_{\text{Image solutions}}\right]$$

where $\bar{H}^*$ is the magnetic field found in Part B.