6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Final- Solutions 2003	
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Problem 1

A

Question: Determine the $\hat{z}$-directed magnetic flux density in the gap of the C-core in terms of the field current i_F, and the parameters of the generator. Make reasonable magnetic circuit approximations, and ignore the flux density sourced by the armature current.

Solution: From Ampere's Law,

$$B_z = \frac{\mu_0 N i_F}{D}$$

B

Question: Determine the self-inductance of the field coil in terms of the parameters of the generator.

Solution:

$$\lambda = NWTB_z \equiv Li \Rightarrow L = \frac{\mu_0 N^2 WT}{D}$$

C

Question: The static terminal relation for the armature takes the form

$$v_A = Ri_A + GUi_F$$

Determine R and G in terms of the parameters of the generator.

Solution: Apply Faraday's Law to the armature circuit and assume perfectly conducting wires.

$$\oint_C E \cdot dC = -\frac{d}{dt}\int_S \underbrace{B \cdot dS}_{zero} = 0$$

$$\underbrace{\int_{(+)}^{(-)}}_{fluid} E_y dy + \underbrace{\int_{(-)}^{(+)}}_{terminals} -\nabla\phi \cdot Jc = 0 \Rightarrow E_y W = v_A$$

Ohm's Law $\Rightarrow \overline{J} = \sigma(\overline{E} + \overline{v} \times \overline{B}) \Rightarrow E_y = \frac{J_y}{\sigma} + vB_z$

$$E_y = \frac{i_A}{\sigma DT} + vB_z$$

$$v_A = \underbrace{\left(\frac{W}{\sigma DT}\right)}_{R} i_A + \underbrace{\left(\frac{\mu_0 NW}{D}\right)}_{G} U i_F$$

D

Question: Determine the mechanical power that is required to pump the fluid through the channel in terms of i_A, i_F, U and the parameters of the generator.

Solution: Force density =

$$\overline{J} \times \overline{B} = J_y B_z \hat{x} = \frac{\mu_0 N i_F i_A}{TD} \hat{x}$$

Power =

$$J_y B_z U \cdot \underbrace{TDW}_{volume} = \frac{\mu_0 NW}{D} i_F i_A U = G i_F i_A U$$

E

Question: The generator is connected such that $i_F = -i_A$ and $v_F = v_A$, in an effort to produce self-excitation. For what range of U will it exhibit such self-excitation? Ignore any armature inductance.

Solution:

$$v_A = R i_A + G U i_F$$
$$v_F = L \frac{di_F}{dt}$$
$$v_F = v_A$$
$$i_F = -i_A$$

Putting everything together,

$$L \frac{di_F}{dt} = -R i_F + G U i_F$$

Self excitation implies

$$GU > R \Rightarrow U > \frac{1}{\mu_0 \sigma N T}$$

Problem 2

A

Question: Find the electric field in the fluid, and in the free space region above the fluid, in terms of $\hat{\phi}_A$, $\hat{\phi}_B$ and $\hat{\phi}_C$.

Solution: In general, $\overline{E} = -\nabla\phi$.

$$\begin{aligned}\overline{E}_{\text{Fluid}} &= \hat{x}\text{Real}\left\{-k\left(\hat{\phi}_A\frac{\cosh(kx)}{\sinh(k\Delta)} - \hat{\phi}_B\frac{\cosh(k(x-\Delta))}{\sinh(k\Delta)}\right)e^{j(\omega t - kz)}\right\}\\ &+ \hat{z}\text{Real}\left\{jk\left(\hat{\phi}_A\frac{\sinh(kx)}{\sinh(k\Delta)} - \hat{\phi}_B\frac{\sinh(k(x-\Delta))}{\sinh(k\Delta)}\right)e^{j(\omega t - kz)}\right\}\\ \overline{E}_{\text{Free Space}} &= \hat{x}\text{Real}\left\{k\hat{\phi}_c e^{-k(x-\Delta)}e^{j(\omega t - kz)}\right\}\\ &+ \hat{z}\text{Real}\left\{jk\hat{\phi}_c e^{-k(x-\Delta)}e^{j(\omega t - kz)}\right\}\end{aligned}$$

B

Question: Using the boundary conditions for an EQS system associated with Gauss' Law and an irrotational $\overline{E}$ field, write three boundary conditions that relate $\hat{\phi}_A$, $\hat{\phi}_B$, $\hat{\phi}_C$ and the surface charge density $\hat{\rho}_S$ to each other, and the parameters of the fluid and the excitation.

Solution:

$$\begin{aligned}\text{At } x = 0 &: \hat{\phi}_B = \hat{v}\\ \text{At } x = \Delta &: \hat{\phi}_A = \hat{\phi}_C\\ \hat{\rho} &= \epsilon_0 k\hat{\phi}_C + \epsilon k\left(\hat{\phi}_A\frac{\cosh(k\Delta)}{\sinh(k\Delta)} - \hat{\phi}_B\frac{1}{\sinh(k\Delta)}\right)\end{aligned}$$

C

Question: Using the boundary condition for charge conservation at $x = \Delta$, write a fourth boundary condition that relates $\hat{\phi}_A$, $\hat{\phi}_B$, $\hat{\phi}_C$ and $\hat{\rho}_S$ to each other, and the parameters of the fluid and the excitation.

Solution:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)\rho = \sigma E_{fluid_x}\big|_{x=\Delta}$$

$$j(\omega - kU)\hat{\rho} = -\sigma k\left(\hat{\phi}_A\frac{\cosh(k\Delta)}{\sinh(k\Delta)} - \hat{\phi}_B\frac{1}{\sinh(k\Delta)}\right)$$

D

Question: Combine the four boundary conditions found in Parts A, B, and C to determine $\hat{\phi}_A$, $\hat{\phi}_B$, $\hat{\phi}_C$ and $\hat{\rho}_S$ in terms of the parameters of the fluid and the excitation.

Solution: Substitute $\hat{\phi}_B = \hat{v}$ and $\hat{\phi}_C = \hat{\phi}_A$.

$$\begin{bmatrix} 1 & -k\left(\epsilon_0 + \epsilon\frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right) \\ j(\omega - kU) & \frac{\sigma k\cosh(k\Delta)}{\sinh(k\Delta)} \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi}_A \end{bmatrix} = \begin{bmatrix} -\epsilon \\ \sigma \end{bmatrix} \frac{k\hat{v}}{\sinh(k\Delta)}$$

$$\begin{bmatrix} \hat{\rho} \\ \hat{\phi}_A \end{bmatrix} = \frac{1}{\frac{\sigma k\cosh(k\Delta)}{\sinh(k\Delta)} + j(\omega - kU)k\left(\epsilon_0 + \epsilon\frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right)} \begin{bmatrix} \frac{\sigma k\cosh(k\Delta)}{\sinh(k\Delta)} & k\left(\epsilon_0 + \epsilon\frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right) \\ -j(\omega - kU) & 1 \end{bmatrix} \begin{bmatrix} \frac{-\epsilon k\hat{v}}{\sinh(k\Delta)} \\ \frac{\sigma k\hat{v}}{\sinh(k\Delta)} \end{bmatrix}$$

$$\hat{\phi}_A = \hat{\phi}_C = \frac{\left(1 + j(\omega - kU)\frac{\epsilon}{\sigma}\right)\hat{v}}{\cosh(k\Delta) + j(\omega - kU)\frac{(\epsilon_0\sinh(k\Delta) + \epsilon\cosh(k\Delta))}{\sigma}}$$

$$\hat{\phi}_B = \hat{v}$$

$$\hat{\rho} = \frac{+\epsilon_0 k\hat{v}}{\cosh(k\Delta) + j(\omega - kU)\frac{(\epsilon_0\sinh(k\Delta) + \epsilon\cosh(k\Delta))}{\sigma}}$$

E

Question: Find the spatially-averaged stress that acts on the free surface of the fluid to pump the fluid in the $\hat{z}$ direction.

Solution: Use the stress tensor to compute the time and space average surface stress that pumps the fluid.

$$\tau = \epsilon_0 E_x E_z\big|_{\text{Free Space at } x=\Delta} - \epsilon E_x E_z\big|_{\text{Fluid at } x=\Delta} = \rho E_z\big|_{x=\Delta}$$

$$\langle\tau\rangle_{z,t} = \frac{1}{2}\text{Real}\left\{jk\hat{\phi}_A\hat{\rho}^*\right\}$$
$$= \frac{-\frac{1}{2}\epsilon_0 k^2|\hat{v}|^2(\omega - kU)\frac{\epsilon}{\sigma}}{\cosh^2(k\Delta) + (\omega - kU)^2\frac{(\epsilon_0\sinh(k\Delta) + \epsilon\cosh(k\Delta))^2}{\sigma^2}}$$

Problem 3

A

Question: For a value of interfacial displacement $\delta(d,t)$, the electric fields E_1 and E_2 in the elastic dielectric and free space are of the form

$$E_1 = A(B + d + \delta(d,t))$$
$$E_2 = C(D + d + \delta(d,t))$$

where A, B, C and D are constants. What are A, B, C, and D?

Solution:

$$E_1(d + \delta(d,t)) + E_2(s - d - \delta(d,t)) = 0 \text{ (Short circuit)}$$

$$\sigma_s = \epsilon_0(E_2 - E_1) = \epsilon_0 E_2 \left(1 + \frac{(s - d - \delta(d,t))}{d + \delta(d,t)}\right) = \frac{\epsilon_0 E_2 s}{d + \delta(d,t)}$$

$$E_2 = \frac{\sigma_s}{\epsilon_0 s}\left(d + \delta(d,t)\right) = C(D + d + \delta(d,t)) \Rightarrow D = 0, C = \frac{\sigma_s}{\epsilon_0 s}$$

$$E_1 = -\frac{E_2(s - d - \delta(d,t))}{d + \delta(d,t)} = -\frac{\sigma_s}{\epsilon_0 s}\left(s - d - \delta(d,t)\right) = A(B + d + \delta(d,t)) \Rightarrow B = -s, A = \frac{\sigma_s}{\epsilon_0 s}$$

B

Question: The electric force per unit area on the interface at $x = d + \delta(d,t)$ takes the form

$$\frac{F_e}{Area} = F + G\delta(d,t)$$

where F and G are constants. Using the results of part (a), determine F and G.

Solution:

$$\begin{aligned} T_{xx}(x = d + \delta(d,t))_+ - T_{xx}(x = d + \delta(d,t))_- &= \frac{\epsilon_0}{2}\left(E_2^2 - E_1^2\right) \\ &= \frac{\epsilon_0}{2}\left(\frac{\sigma_s}{\epsilon_0 s}\right)^2 \left[(d + \delta(d,t))^2 - (s - (d + \delta(d,t))^2)\right] \\ &= \frac{\epsilon_0}{2}\left(\frac{\sigma_s}{\epsilon_0 s}\right)^2 \left[-s^2 + 2(d + \delta(d,t))s\right] \end{aligned}$$

Another way:

$$\begin{aligned} \frac{F_e}{Area} &= \frac{1}{2}\sigma_s(E_1 + E_2) \\ &= \frac{1}{2}\sigma_s\left(\frac{\sigma_s}{\epsilon_0 s}\right)\left[d + \delta(d,t) - s + (d + \delta(d,t))\right] \\ &= \frac{1}{2}\frac{\sigma_s^2}{\epsilon_0 s}\left[2d + 2\delta(d,t) - s\right] \\ &= F + G\delta(d,t) \\ F &= \frac{1}{2}\frac{\sigma_s^2}{\epsilon_0 s}(2d - s), G = \frac{\sigma_s^2}{\epsilon_0 s} \end{aligned}$$

C

Question: What is the steady state elastic displacement $\delta_{ss}(x)$?

Solution:

$$E\frac{\partial^2 \delta_{ss}}{\partial x^2} = 0 \Rightarrow \delta_{ss} = ax + b$$

$$\delta_{ss}(x=0) = b = 0$$

$$\text{at } x = d + \delta_{ss}(d) : -E\, \frac{\partial \delta_{ss}}{\partial x}\bigg|_{x=d} + \frac{F_e}{Area} = 0$$

$$-Ea + \frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(2d - s + 2ad) = 0$$

$$a\left[\frac{\sigma_s^2 d}{\epsilon_0 s} - E\right] = \frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s - 2d)$$

$$a = \frac{1}{2}\frac{\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s-2d)}{\frac{\sigma_s^2 d}{\epsilon_0 s} - E}$$

$$\delta_{ss}(x) = ax = \frac{1}{2}\frac{\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s-2d)x}{\frac{\sigma_s^2 d}{\epsilon_0 s} - E}$$

D

Question: Now assume that the system is slightly perturbed so that the elastic displacement is of the form

$$\delta(x,t) = \delta_{ss}(x) + \delta'(x,t)$$

Take the general form of $\delta'(x,t)$ to be

$$\delta'(x,t) = Re\left[\hat{\delta}(x)e^{j\omega t}\right]$$

and find the spatial dependence for $\hat{\delta}(x)$.

Solution:

$$\delta(x,t) = \delta_{ss}(x) + \delta'(x,t)$$

$$\rho\frac{\partial^2 \delta'}{\partial t^2} = E\frac{\partial^2 \delta'}{\partial x^2}; \delta'(x,t) = Re[\hat{\delta}(x)e^{j\omega t}]$$

$$\hat{\delta}(x) = A\sin(kx) + B\cos(kx)$$

$$\hat{\delta}(x=0) = 0 = B \Rightarrow \hat{\delta}(x) = A\sin(kx)$$

E

Question: The system natural frequencies take the form

$$\tan(kd) = Hkd, k = \omega\sqrt{\frac{\rho}{E}}$$

where H is a constant. What is H?

Solution:

$$-E \left.\frac{\partial \hat{\delta}}{\partial x}\right|_{x=d} + \frac{\sigma_s^2}{\epsilon_0 s}\hat{\delta}(d) = 0$$

$$-Ek\cos(kd) + \frac{\sigma_s^2}{\epsilon_0 s}\sin(kd) = 0$$

$$\tan(kd) = \frac{E(kd)}{\sigma_s^2 d}\epsilon_0 s = H(kd)$$

$$H = \frac{E\epsilon_0 s}{\sigma_s^2 d}$$

F

Question: At what value of σ_s is the system first unstable?

Solution: For $H > 1$, solutions have k and ω real $\Rightarrow$ stable.
For $H < 1$, solutions have k and ω imaginary $\Rightarrow$ Unstable

$$k = jk_i$$

$$\tanh(k_i d) = H(k_i d)$$

Critical value of $\sigma_s : H = 1 \Rightarrow \sigma_s = \left[\frac{\epsilon_o s E}{d}\right]^{\frac{1}{2}}$

Problem 4

A

Question: The switch at $z = 0$ is open for $t < 0$ and the voltage source has been connected to the transmission line for a long time so that all transient waves have died away and the transmission line voltage and current are in the DC steady state. What are the steady-state voltage and current on the transmission line for $t < 0$?

Solution:

$$i = \frac{V_0}{R_S + 2Z_0}, v = \frac{2Z_0 V_0}{R_S + 2Z_0}$$

B

Question: With the transmission line in the DC steady state of part (a), the switch at $z = 0$ is closed for all time $t > 0$. The resulting voltage and current transient waves on the transmission line can be written as

$$v(z,t) = V_+\left(t - \frac{z}{c}\right) + V_-\left(t + \frac{z}{c}\right); i(z,t)Z_0 = V_+\left(t - \frac{z}{c}\right) - V_-\left(t + \frac{z}{c}\right)$$

What are $V_+\left(t - \frac{z}{c}\right)$ and $V_-\left(t + \frac{z}{c}\right)$ at times $t = 0$ and $t = \infty$?

Solution:

$t = 0:$

$$V_+ + V_- = \frac{2Z_0V_0}{R_S + 2Z_0}$$
$$V_+ - V_- = \frac{V_0Z_0}{R_S + 2Z_0}$$
$$V_+ = \frac{1}{2}\frac{V_0(3Z_0)}{R_S + 2Z_0}$$
$$V_- = \frac{1}{2}\frac{V_0Z_0}{R_S + 2Z_0}$$

$t = \infty:$

$$V_+ + V_- = V_0\frac{Z_0}{Z_0 + R_S}$$
$$V_+ - V_- = \frac{V_0Z_0}{R_S + Z_0}$$
$$V_+ = \frac{V_0Z_0}{R_S + Z_0}$$
$$V_- = 0$$

C

Question: **At $z = 0$, with the switch closed for $t > 0$, calculate**

$$\left.\frac{V_-}{V_+}\right|_{z=0}$$

Solution:

$$\left.\frac{V_-}{V_+}\right|_{z=0} = 0 \text{ matched end}$$

D

Question: **At $z = -l$ for $t > 0$, the positive z directed wave is of the form**

$$V_+|_{z=-l} = A + BV_-|_{z=-l}$$

What are A and B?

Solution: At $z = -l:$

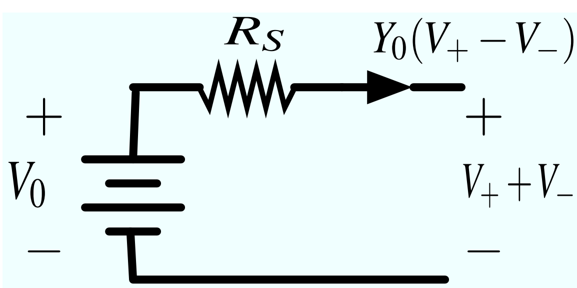

Figure 1: Equivalent circuit at $z = -l$. (Image by MIT OpenCourseWare.)

$$-V_0 + Y_0R_S(V_+ - V_-) + V_+ + V_- = 0$$
$$V_+(Y_0R_S + 1) + V_-(1 - Y_0R_S) = V_0$$

$$\begin{aligned}V_+ &= \frac{V_0}{1+Y_0R_S} + \frac{V_-(Y_0R_S-1)}{Y_0R_S+1}\\ &= \frac{V_0}{1+\frac{R_S}{Z_0}} + \frac{V_-(R_S-Z_0)}{R_S+Z_0}\\ &= A + BV_-\end{aligned}$$

$$A = \frac{V_0}{1+\frac{R_S}{Z_0}}, B = \frac{R_S-Z_0}{R_S+Z_0} = \underbrace{\Gamma_S}_{\text{source reflection coefficient}}$$

E

Question: The wave trajectories in $z-t$ space demarcate the solution regions as shown in the figure (see exam). Consider the case where $R_S = 0$. What are V_+ and V_- in regions $1, 2, 3, 4, 5, 6, 7, 8, 9$?

Solution:

$$V_+\big|_{t=0} = \frac{3}{2}\frac{V_0}{2} = \frac{3}{4}V_0$$

$$V_-\big|_{t=0} = \frac{1}{2}\frac{V_0}{2} = \frac{V_0}{4}$$

At $x = 0, V_- = 0$

At $x = -l, V_+ = V_0 - V_-$

Region 1: $V_+ = \frac{3}{4}V_0, V_- = \frac{V_0}{4}$

Region 2: $V_+ = \frac{3}{4}V_0, V_- = 0$

Region 3: $V_+ = V_0 - \frac{V_0}{4} = \frac{3}{4}V_0, V_- = \frac{V_0}{4}$

Region 4: $V_+ = \frac{3}{4}V_0, V_- = 0$

Region 5: $V_+ = \frac{3}{4}V_0, V_- = 0$

Region 6: $V_+ = V_0, V_- = 0$

Region 7: $V_+ = V_0, V_- = 0$

Region 8: $V_+ = V_0, V_- = 0$

Region 9: $V_+ = V_0, V_- = 0$

F

Question: Give a labeled plot of $v(z = -\frac{l}{4}, t)$ and $i(z = -\frac{l}{4}, t)Z_0$ for time $t \geq 0$

Solution:

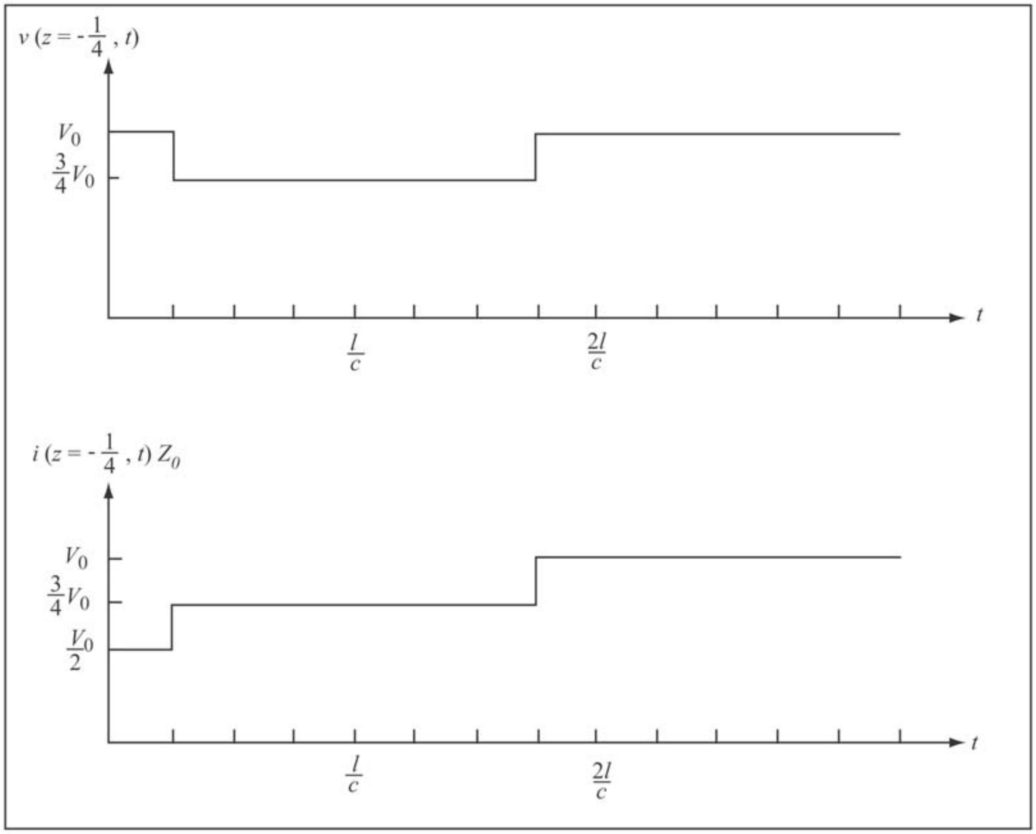

Figure 2: Solution to Part F. A labeled plot of $v(z = -\frac{l}{4}, t)$ and $i(z = -\frac{l}{4}, t)Z_0$ for time $t \geq 0$ (Image by MIT OpenCourseWare.)

Problem 5

A

Question: In the long wavelength limit, what is the electric force per unit length on the string to first order in $\xi(x,t)$ when $\xi(x,t) \ll s$?

Solution: Using the method of images,

$$\begin{aligned} f = \lambda E &= \frac{\lambda^2}{2\pi\epsilon_0(2)(s - \xi(x,t))} \\ &= \frac{\lambda^2}{4\pi\epsilon_0 s\left(1 - \frac{\xi(x,t)}{s}\right)} \\ &\approx \frac{\lambda^2}{4\pi\epsilon_0 s}\left(1 + \frac{\xi(x,t)}{s}\right) \end{aligned}$$

B

Question: For what value of λ will the string have an equilibrium with $\xi(x,t) = 0$?

Solution:

$$mg = \frac{\lambda^2}{4\pi\epsilon_0 s} \text{ when } \xi(x,t) = 0$$
$$\lambda = \sqrt{4\pi\epsilon_0 smg}$$

C

Question: For small signal wave solutions of the form

$$\xi(x,t) = Re\left[\hat{\xi}e^{j(\omega t - kx)}\right]$$

what is the $\omega - k$ dispersion relation?

Solution:

$$m\frac{\partial^2\xi}{\partial t^2} = T\frac{\partial^2\xi}{\partial x^2} - mg + \frac{\lambda^2}{4\pi\epsilon_0 s}\left(1 + \frac{\xi(x,t)}{s}\right)$$
$$\frac{\partial^2\xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2\xi}{\partial x^2} + \frac{\lambda^2}{4\pi\epsilon_0 s^2 m}\xi(x,t)$$
$$\xi(x,t) = Re\left[\hat{\xi}e^{j(\omega t - kx)}\right]$$
$$-\omega^2 = -k^2 v_p^2 + \omega_c^2; v_p = \sqrt{\frac{T}{m}}, \omega_c^2 = \frac{\lambda^2}{4\pi\epsilon_0 s^2 m}$$
$$\omega^2 = k^2 v_p^2 - \omega_c^2$$

D

Question: Applying the boundary conditions at $x = 0$ and $x = -l$, the allowed values of k can be obtained from the transcendental relationship of the form

$$\tan(kl) = Ckl$$

where C is a constant. What is C?

Solution:

$$\xi(x,t) = Re\left[\hat{\xi}(x)e^{j\omega t}\right]$$
$$\hat{\xi}(x) = A\sin(kx) + B\cos(kx), k = \frac{\omega^2 + \omega_c^2}{v_p^2}$$
$$\xi(x=0) = 0 = B$$

$$T\left.\frac{\partial\xi}{\partial x}\right|_{x=-l} - K\xi(x=-l,t) = 0$$
$$T\not{A}k\cos(kl) + K\not{A}\sin(kl) = 0$$
$$\tan(kl) = -\frac{T}{K}k = C(kl) \Rightarrow C = -\frac{T}{Kl}$$

E

Question: If the spring constant is zero, $K = 0$, what are the solutions for wavenumber k from part (d)?

Solution:

$$K = 0 \Rightarrow \tan(kl) = \infty \Rightarrow kl = (2n+1)\frac{\pi}{2}, n = 0, 1, 2, \ldots$$

F

Question: For the conditions of part (e), what is the maximum string mass per unit length m that can be stably supported with $\xi(x,t) = 0$?

Solution:

$$(kl)_{\min} = \frac{\pi}{2}$$

For stability $\omega_c^2 < (kv_p)^2$.

$$\frac{\lambda^2}{4\pi\epsilon_0 s^2 m} < \left(\frac{\pi}{2l}\right)^2 v_p^2$$
$$\frac{\lambda^2}{4\pi\epsilon_0 s^2 \cancel{m}} < \left(\frac{\pi}{2l}\right)^2 \frac{T}{\cancel{m}}$$

From part (b), $\frac{\lambda^2}{4\pi\epsilon_0 s} = mg \Rightarrow$

$$\frac{mg}{s} < \left(\frac{\pi}{2l}\right)^2 T$$
$$m < \left(\frac{\pi}{2l}\right)^2 s\frac{T}{g}$$