MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF COMPUTER SCIENCE AND ELECTRICAL ENGINEERING

6.801/6.866 MACHINE VISION

Handed out: 2004 Nov 18th **Due on: 2003 Nov 30th**

Problem 1: The motion field is particularly simple when a camera is moving without rotation in a fixed environment. All parts of the image then appear to be streaming away from the "focus of expansion"—the projection of the translational motion vector $\mathbf{t}$ into the image plane.

(a) Show that the focus of expansion is the vanishing point of the family of parallel lines ($\mathbf{r} = \mathbf{r}_0 + k\mathbf{t}$) in the direction of translational motion.

(b) Show that, the location of the focus of expansion (x_0, y_0) is given by

$$\frac{1}{f}(x_0, y_0) = \frac{1}{W}(U, V)$$

if the instantaneous translational velocity is $\mathbf{t} = (U, V, W)^T$.

(c) Show that

$$(x - x_0)E_x + (y - y_0)E_y = 0$$

at "critical" points, that is where $E_t = 0$.

(d) Estimate the position of the FOE by minimizing

$$\sum_{i=1}^{n}\bigl((x_i - x_0)E_{x_i} + (y_i - y_0)E_{y_i}\bigr)^2,$$

where (x_i, y_i) are the positions of n "critical" points in the image, while (E_{x_i}, E_{y_i}) are the brightness gradients at these points.

Problem 2: One method for finding the orientation of an object described by an extended Gaussian image is to find the directions of axes where the inertia has stationary values (minima, maxima, and saddle points).

(a) Show that the inertia about the axis $\boldsymbol{\omega}$ through the origin can be written in the form

$$\iint G(\eta, \xi)(\mathbf{r} \cdot \mathbf{r}) \cos\eta \, d\eta \, d\xi - \iint G(\eta, \xi)(\mathbf{r} \cdot \boldsymbol{\omega})^2 \cos\eta \, d\eta \, d\xi$$

subject to the constraint that $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1$, where $G(\eta, \xi)$ is the density on the unit sphere, and $\mathbf{r} = (\cos\eta\cos\xi, \cos\eta\sin\xi, \sin\eta)^T$. Where does the $\cos\eta$ in the integrands come from?

(b) Note that the first integral above does not depend on the direction $\boldsymbol{\omega}$ of the axis. Show that the problem reduces to finding stationary values of the second integral above, subject to the constraint $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1$, or equivalently, finding stationary values of the Raleigh ratio

$$\boldsymbol{\omega}^T \left(\iint G(\eta, \xi)(\mathbf{r}\,\mathbf{r}^T) \cos\eta \, d\eta \, d\xi \right) \boldsymbol{\omega} \Big/ \boldsymbol{\omega}^T \boldsymbol{\omega}$$

(c) How are the "inertia axes" related to the symmetric 3×3 matrix in the formula above? Is this method practical when the experimental data was obtained using the photometric stereo method?

Problem 3: Consider the solid of revolution obtained by rotating an ellipse about its minor axis. Let the major and minor axes have length a and b. Suppose that the generating ellipse lies in the x-y plane, that the axis of rotation is the y-axis, and that the ellipse can be parameterized as follows:

$$x = a\cos\theta \qquad \text{and} \qquad y = b\sin\theta.$$

(a) Show that

$$\frac{ds}{d\theta} = \sqrt{b^2\cos^2\theta + a^2\sin^2\theta},$$

where s is arc length along the generating ellipse.

(b) Show that a unit normal to the generating ellipse at the point $(a\cos\theta, b\sin\theta)^T$ is given by

$$\frac{1}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}(b\cos\theta, a\sin\theta)^T$$

(c) The unit normal can also be written $(\cos\eta, \sin\eta)^T$ in terms of the angle η between the normal vector and the x-axis. Show that

$$\frac{b^2\cos^2\theta + a^2\sin^2\theta}{ab} = \frac{ab}{a^2\cos^2\eta + b^2\sin^2\eta}$$

Show that $\tan\eta = (a/b)\tan\theta$ and that

$$\frac{d\eta}{d\theta} = \frac{ab}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{a^2\cos^2\eta + b^2\sin^2\eta}{ab}.$$

(d) Give the curvature K_G of the generating curve first as a function of θ — and then as a function of the angle of the normal η.

(e) Finally, show that the Gaussian curvature of the solid of revolution can be written in the form

$$K = \frac{1}{a^2}\left(\frac{a^2\cos^2\eta + b^2\sin^2\eta)}{ab}\right)^2.$$

What is the extended Gaussian image $G(\eta, \xi)$ of this object? What are the extreme values of $G(\eta, \xi)$? Verify these result for the special case $a = b$.

Problem 4:

(a) In order to construct an orientation histogram of EGI, we find it convenient to divide the surface of the sphere in some regular fashion. Projections of the Platonic and Archimedean solids and their duals are useful for this purpose. A geodesic sphere of order n may be constructed by dividing each of the triangular faces of an icosahedron into n^2 triangular facets. How many cells does the dual of this object have? How many of these cells are pentagons? How many are hexagons?

(b) Consider a planar curve with the curious property that its curvature is proportional to the distance from some line. That is $K_G = Ar$, where $K_G = d\eta/ds$, A is a constant, and r is the perpendicular distance from the line. At the points where the curve touches the line it is perpendicular to the line. Now imagine spinning the curve about the line. Compute the extended Gaussian image (EGI) of the resulting solid of revolution. Does this correspond to the EGI of an object we discussed in class?

Problem 5: Consider the equation

$$\frac{d\mathring{q}}{dt} = \mathring{\omega}\mathring{q}$$

where $\mathring{\omega}$ is a purely imaginary quaternion (that is, its real part is zero).

(a) The series expansion of $e^{\omega t}$ can be used to give meaning to the expression $e^{\mathring{\omega} t}$. Show that formally

$$\mathring{q} = e^{\mathring{\omega} t}\mathring{q}_0$$

is a solution of the above differential equation.

(b) Show that

$$\mathring{\omega}\,\mathring{\omega} = -(\mathring{\omega} \cdot \mathring{\omega})\mathring{e}$$

where $\mathring{e}$ is the unit quaternion with real part 1, and zero imaginary part. Hint: Use the fact that $\mathring{\omega}$ has zero real part. Give expressions for $\mathring{\omega}^{2n}$ and $\mathring{\omega}^{2n+1}$.

(c) Write an expression for $e^{\mathring{\omega}}$ in terms of $\mathring{e}$, $\mathring{\omega}$, sin and cos of some angle. Hint: consider the power series for cos and sin.

(d) Now suppose that $\mathring{\omega} = (1/2)(0, \omega\hat{\boldsymbol{\omega}})$. Rewrite $e^{\mathring{\omega}t}$ in terms of $\cos(\omega t/2)$ and $\sin(\omega t/2)$. Show that

$$\mathring{q} = \mathring{r}\mathring{q}_0$$

where $\mathring{r}$ is the quaternion representing the rotation through an angle $\theta = \omega t$ about an axis $\hat{\boldsymbol{\omega}}$.