Massachusetts Institute of Technology

Department of Computer Science and Electrical Engineering

6.801/6.866 Machine Vision HWP 4 Solutions

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PROBLEM 1 (10 points)

The integralbeing minimized is of the form

$$\iint_D F(u, v, u_x, u_y, v_x, v_y)\, dx\, dy$$

where

$$F(u, v, u_x, u_y, v_x, v_y) = (uE_x + vE_y + E_t)^2 + \lambda(u_x^2 + u_y^2 + v_x^2 + v_y^2)$$

we have the natural boundary conditions

$$F_{u_x} \frac{dy}{ds} = F_{u_y} \frac{dx}{ds}$$
$$F_{v_x} \frac{dy}{ds} = F_{v_y} \frac{dx}{ds}$$

i.e.,

$$2\lambda u_x \frac{dy}{ds} = 2\lambda u_y \frac{dx}{ds}$$
$$2\lambda v_x \frac{dy}{ds} = 2\lambda v_y \frac{dx}{ds}$$

$\Longrightarrow$

$$(u_x, u_y) \cdot (\frac{dy}{ds}, -\frac{dx}{ds}) = 0$$
$$(v_x, v_y) \cdot (\frac{dy}{ds}, -\frac{dx}{ds}) = 0$$

Both gradient directions (u_x, u_y) and (v_x, v_y) are perpendicular to the vector $(\frac{dy}{ds}, -\frac{dx}{ds})$. Since $n = (\frac{dy}{ds}, -\frac{dx}{ds})$ is the normal to the curve along occluding boundary D, we can further conclude that the gradient direction are parallel to the tangent direction of boundary D.

PROBLEM 2 (20 points)

We would like to consider the general case (as shown in figure) that a sphere with radius R is centered at (X_0, Y_0, Z_0) whose corresponding point is at (x_0, y_0).

Let (X, Y, Z) be the cooridnate for point P on 3D sphere. Its corresponding point p in 2D image is at (x, y). The sphere is rotating around its axis that is parallel to the $y-$ axis with angular velocity ω. Without losing generality, we can assume that the rotation is counter-clockwise, thus the visible motion in image is $u(x, y) \leq 0$.

Specifically, for point P in figure ,

$$\begin{aligned}
\tfrac{dX}{dt} &= \omega r \sin\theta \\
&= \omega\sqrt{R^2 - (X - X_0)^2 - (Y - Y_0)^2} \\
\tfrac{dY}{dt} &= 0
\end{aligned}$$

For parallel projection, we have the following relationship:

$$\begin{aligned}
x &= (f/Z_0)X \\
y &= (f/Z_0)Y \\
x_0 &= (f/Z_0)X_0 \\
y_0 &= (f/Z_0)Y_0
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x, y) &= \frac{dx}{dt} \\
&= (f/Z_0)\frac{dX}{dt} \\
&= (f/Z_0)\omega\sqrt{R^2 - (X - X_0)^2 - (Y - Y_0)^2} \\
&= \omega\sqrt{(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2} \\
v(x, y) &= \frac{dy}{dt} \\
&= (f/Z_0)\frac{dY}{dt} \\
&= 0
\end{aligned}$$

Please note that:

1. The motion depends only on $(X - X_0)^2 + (Y - Y_0)^2$ which is rotationally symmetric.
2. fR/Z_0 is the radius of the circular projection corresponding to the sphere.

The measurement of unsmoothness is based on the following calculation:

$$\begin{aligned}
& u_x^2 + u_y^2 + v_x^2 + v_y^2 \\
= \ & \left(\frac{-\omega(x - x_0)}{\sqrt{(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2}}\right)^2 + \left(\frac{-\omega(y - y_0)}{\sqrt{(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2}}\right)^2 \\
= \ & \frac{\omega^2((x - x_0)^2 + (y - y_0)^2)}{(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2}
\end{aligned}$$

The "unsmoothness" goes to infinity when

$$(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2 = 0$$

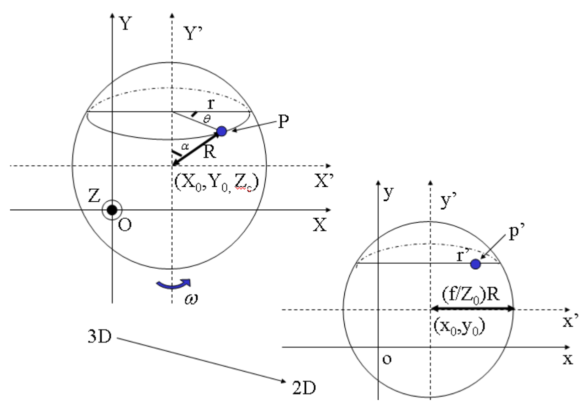
i.e., at the circular occluding boundary of sphere projection.
Specifically,
1. u_x goes to infinity at the occluding boundary except when $x = x_0$.
2. u_y goes to infinity at the occluding boundary except when $y = y_0$.
For our specific case, $X_0 = Y_0 = 0$, i.e., $x_0 = y_0 = 0$, thus we have:

$$\begin{aligned} u(x,y) &= \omega\sqrt{(fR/Z_0)^2 - (x)^2 - (y)^2} \\ v(x,y) &= 0 \end{aligned}$$

The "unsmoothness" goes to infinity when

$$(fR/Z_0)^2 - x^2 - y^2 = 0.$$

$$(fR/Z_0)^2 - (x - x_0)^2 - (y - y_0)^2 = 0.$$



(a)

Figure 1: Figure for Problem 5.

Note: In our problem, the origin of the 3D is not necessarily at the center of sphere, i.e., $Z_0 \neg 0$. The sphere rotates around line determined by $Z = Z_0$ and $X = 0$ instead of y axis. If people do assume the center of sphere is at the origin, you should be aware why you can do so. For orthographic projection, the 2D projection of the sphere is the same when the center moves long the z axis. Besides, the motion described in this problem is also only determined by its (X, Y) coordinate.

PROBLEM 3 (20 points)

We have three unknown functions z, p, q in two variables (x, y):

$$\iint_D F(p,q,z,p_x,q_x,z_x,p_y,q_y,z_y)\,dx\,dy$$

The three Eular equations are:
For z,

$$\begin{aligned} F_z - \frac{\partial}{\partial x}F_{z_x} - \frac{\partial}{\partial y}F_{z_y} &= 0 \quad \Longrightarrow \\ 0 - \frac{\partial}{\partial x}(2\mu(z_x - p)) - \frac{\partial}{\partial y}(2\mu(z_y - p)) &= 0 \quad \Longrightarrow \end{aligned}$$

$$\boxed{z_{xx} + z_{yy} = p_x + q_y}$$

For p,

$$\begin{aligned} F_p - \frac{\partial}{\partial x}F_{p_x} - \frac{\partial}{\partial y}F_{p_y} &= 0 \quad \Longrightarrow \\ \Big(-2R_p(E(x,y) - R(p,q)) - 2\mu(z_x - p)\Big) - \frac{\partial}{\partial x}(2\lambda p_x) - \frac{\partial}{\partial y}(2\lambda p_y) &= 0 \quad \Longrightarrow \end{aligned}$$

$$\boxed{R_p(E(x,y) - R(p,q)) + \mu(z_x - p) + \lambda(p_{xx} + p_{yy}) = 0}$$

For q,

$$\begin{aligned} F_q - \frac{\partial}{\partial x}F_{q_x} - \frac{\partial}{\partial y}F_{q_y} &= 0 \quad \Longrightarrow \\ \Big(-2R_q(E(x,y) - R(p,q)) - 2\mu(z_y - q)\Big) - \frac{\partial}{\partial x}(2\lambda q_x) - \frac{\partial}{\partial y}(2\lambda q_y) &= 0 \quad \Longrightarrow \end{aligned}$$

$$\boxed{R_q(E(x,y) - R(p,q)) + \mu(z_y - q) + \lambda(q_{xx} + q_{yy}) = 0}$$

PROBLEM 4 (20 points)

If the "unsmoothness" is defined by

$$\iint F(z,z_x,z_y)\,dx\,dy = \iint (z_x^2 + z_y^2)\,dx\,dy$$

we have Eular equation:

$$F_z - \frac{\partial}{\partial x}F_{z_x} - \frac{\partial}{\partial y}F_{z_y} = 0 \quad \Longrightarrow$$

$$0 - 2z_{xx} - 2z_{yy} = 0 \quad \Longrightarrow$$

$$\boxed{z_{xx} + z_{yy} = 0}$$

As in Prob.1, the natural boundary condition is as followed:

$$(z_x, z_y) \cdot \Big(\frac{dy}{ds}, -\frac{dx}{ds}\Big) = 0 \qquad \text{on } \delta \text{D}$$

If the "unsmoothness" is defined by

$$\iint F(z, z_{xx}, z_{yy})\, dx\, dy = \iint (z_{xx} + z_{yy})^2\, dx\, dy$$

we have Eular equation:

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} + \frac{\partial^2}{\partial x^2} F_{z_{xx}} + \frac{\partial^2}{\partial x \partial y} F_{z_{xy}} + \frac{\partial^2}{\partial y^2} F_{z_{yy}} = 0 \qquad \Longrightarrow$$

$$\frac{\partial^2}{\partial x^2} F_{z_{xx}} + \frac{\partial^2}{\partial y^2} F_{z_{yy}} = 0 \qquad \Longrightarrow$$

$$2\frac{\partial^2}{\partial x^2}(z_{xx} + z_{yy}) + 2\frac{\partial^2}{\partial y^2}(z_{xx} + z_{yy}) = 0 \qquad \Longrightarrow$$

$$\boxed{z_{xxxx} + 2z_{xxyy} + z_{yyyy} = 0}$$

Note: The Eular equation for this case can be derived by following the same step as in Appendix. (Revised based on derivation provided by Jiawen Chen and Carrick Detweiler.) For

$$I = \iint_D F(x, y, f, f_x, f_y, f_{xx}, f_{xy}, f_{yy})\, dx\, dy$$

Let $\eta(x)$ be a test function. If we add $\epsilon\eta(x)$ to $f(x)$, we expect that the integral will experience only small variation. Specifically, we expect to have

$$\frac{dI(\epsilon)}{d\epsilon} = \frac{d}{d\epsilon} \iint_D F(x, y, f + \epsilon\eta, f_x + \epsilon\eta_x, f_y + \epsilon\eta_y, f_{xx} + \epsilon\eta_{xx}, f_{yy} + \epsilon\eta_{yy})\, dx\, dy = 0$$

Since we have

$$\begin{aligned} & F(x, y, f + \epsilon\eta, f_x + \epsilon\eta_x, f_y + \epsilon\eta_y, f_{xx} + \epsilon\eta_{xx}, f_{yy} + \epsilon\eta_{yy}) \\ = \;& F(x, y, f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}) + \epsilon F_f \eta \\ + \;& \epsilon F_{f_x}\eta_x + \epsilon F_{f_y}\eta_y \\ + \;& \epsilon F_{f_{xx}}\eta_{xx} + \epsilon F_{f_{xy}}\eta_{xy} + \epsilon F_{f_{yy}}\eta_{yy} + e \end{aligned}$$

we expect that the following equation holds.

$$\iint_D \Big(F_f\eta + F_{f_x}\eta_x + F_{f_y}\eta_y + F_{f_{xx}}\eta_{xx} + F_{f_{xy}}\eta_{xy} + F_{f_{yy}}\eta_{yy}\Big)\, dx\, dy = 0 \qquad (1)$$

Gauss's integral theorem tells us

$$\int_{\partial D}(Qdy - Pdx) = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\, dy$$

Now we will apply Gauss's integral theorem twice. For our first application, we let

$$\begin{aligned} Q &= \eta_x F_{f_{xx}} + \frac{1}{2}\eta_y F_{f_{xy}} \\ P &= \eta_y F_{f_{yy}} + \frac{1}{2}\eta_x F_{f_{xy}} \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{\partial D}\Big(\Big(\eta_x F_{f_{xx}} + \frac{1}{2}\eta_y F_{f_{xy}}\Big)dy - \Big(\eta_y F_{f_{yy}} + \frac{1}{2}\eta_x F_{f_{xy}}\Big)dx\Big) \\ = & \iint_D \Big(\frac{\partial}{\partial x}\Big(\eta_x F_{f_{xx}} + \frac{1}{2}\eta_y F_{f_{xy}}\Big) + \frac{\partial}{\partial y}\Big(\eta_y F_{f_{yy}} + \frac{1}{2}\eta_x F_{f_{xy}}\Big)\Big)\, dx\, dy \\ = & \iint_D \Big(\eta_{xx} F_{f_{xx}} + \eta_{xy} F_{f_{xy}} + \eta_{yy} F_{f_{yy}}\Big)\, dx\, dy \\ + & \iint_D \Big(\eta_x\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big) + \eta_y\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)\Big)\, dx\, dy \end{aligned}$$

Given the boundary condition, the term on the right must be zero, so that

$$\begin{aligned} & \iint_D \Big(\eta_{xx} F_{f_{xx}} + \eta_{xy} F_{f_{xy}} + \eta_{yy} F_{f_{yy}}\Big)\, dx\, dy \\ = & -\iint_D \Big(\eta_x\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big) + \eta_y\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)\Big)\, dx\, dy \end{aligned} \tag{2}$$

Applying Guass's integral theorem again with

$$Q = \eta\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big) P = \eta\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)$$

we have

$$\begin{aligned} & \int_{\partial D}\Big(\eta\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big)dy - \eta\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)dx\Big) \\ = & \iint_D \Big(\frac{\partial}{\partial x}\Big(\eta\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big)\Big) + \frac{\partial}{\partial y}\Big(\eta\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)\Big)\Big)\, dx\, dy \\ = & \iint_D \Big(\eta_x\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big) + \eta_y\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)\Big)\, dx\, dy \\ + & \iint_D \Big(\eta\Big(\frac{\partial^2}{\partial x^2}F_{f_{xx}} + \frac{\partial^2}{\partial x \partial y}F_{f_{xy}}) + \frac{\partial^2}{\partial y^2}F_{f_{yy}}\Big)\Big)\, dx\, dy \end{aligned}$$

Given the boundary condition, the term on the right must be zero, so that

$$-\iint_D \left(\eta_x\Big(\frac{\partial}{\partial x}F_{f_{xx}} + \frac{1}{2}\frac{\partial}{\partial y}F_{f_{xy}}\Big) + \eta_y\Big(\frac{\partial}{\partial y}F_{f_{yy}} + \frac{1}{2}\frac{\partial}{\partial x}F_{f_{xy}}\Big)\right) dx\, dy \tag{3}$$
$$= \iint_D \left(\eta\Big(\frac{\partial^2}{\partial x^2}F_{f_{xx}} + \frac{\partial^2}{\partial x \partial y}F_{f_{xy}} + \frac{\partial^2}{\partial y^2}F_{f_{yy}}\Big)\right) dx\, dy$$

Compare equation [2][3], we conclude that

$$\boxed{\iint_D \left(\eta(\tfrac{\partial^2}{\partial x^2}F_{f_{xx}} + \tfrac{\partial^2}{\partial x \partial y}F_{f_{xy}} + \tfrac{\partial^2}{\partial y^2}F_{f_{yy}})\right) dx\, dy = \iint_D \left(\eta_{xx}F_{f_{xx}} + \eta_{xy}F_{f_{xy}} + \eta_{yy}F_{f_{yy}}\right) dx\, dy} \tag{4}$$

In the textbook, we also proved that

$$\boxed{\iint_D \left(\eta\tfrac{\partial}{\partial x}F_{f_x} + \eta\tfrac{\partial}{\partial y}F_{f_y}\right) dx\, dy = -\iint_D \left(\eta_x F_{f_x} + \eta_y F_{f_y}\right) dx\, dy} \tag{5}$$

Based on equation[4][5], to make equation [1] hold, we must have:

$$\iint_D \eta\Big(F_f - \frac{\partial}{\partial x}F_{f_x} - \frac{\partial}{\partial y}F_{f_y} + \frac{\partial^2}{\partial x^2}F_{f_{xx}} + \frac{\partial^2}{\partial x \partial y}F_{f_{xy}} + \frac{\partial^2}{\partial y^2}F_{f_{yy}}\Big) dx\, dy = 0$$

must hold for all possible test functions η, which implies that

$$F_f - \frac{\partial}{\partial x}F_{f_x} - \frac{\partial}{\partial y}F_{f_y} + \frac{\partial^2}{\partial x^2}F_{f_{xx}} + \frac{\partial^2}{\partial x \partial y}F_{f_{xy}} + \frac{\partial^2}{\partial y^2}F_{f_{yy}} = 0$$

For our case, it is

$$\frac{\partial^2}{\partial x^2}F_{f_{xx}} + \frac{\partial^2}{\partial y^2}F_{f_{yy}} = 0$$

which is the conclusion that we have used.

PROBLEM 5 (30 points)

This is a problem related to compensating image motion recovery for finite scan-out speed. To estimate the velocity of the image, we need to minimize the following cost

$$\iint (uE'_x + vE'_y + E'_t)^2$$

where E'_x, E'_y, E'_t are computed based on scanned images instead of underlying images.

Part a (6 points).

The image velocity will be correctly estimated (as zero).

Note that when the image does not move from frame to frame, pixel value will remain to be the same from one frame to the next. Thus we can conclude that there is no motion involved.

Part b (6 points).

The image velocity will be correctly estimated.

Note that each row is read all at once. Thus the motion for different rows of the image being read out and for the underlying time-varying image is the **same** when the real image moves strictly in the x-direction. We can correctly estimate the motion no matter which portion of image we use.

In another word, when $v = 0$, we only need E'_x to estimate u. Since $E'_x = E_x$ holds, our estimation would be accurate.

Part c (6 points).

The image velocity will not be correctly estimated.

Consider an extreme case. When underlying image moves downward at the same speed of scanning, then every row of read-out image would be equal to the first scanned row, which would not provide any information of true image motion.

Specifically consider image row at vertical location l in the underlying image. In the first scanned image, its corresponding location is at row:

$$l - \frac{l}{H/T}v.$$

In the second scanned image, its corresponding location is at row

$$l - Tv - \frac{(l - Tv)}{H/T}v.$$

The vertical shift is

$$-Tv + \frac{Tv}{H/T}v = -Tv(1 - \frac{Hv}{T})$$

which is different from the vertical shift is $-Tv$ for normal scanning case.

"Compression" effect

When the underlying image moves strictly in the y-direction with v, the scanning speed changes from initial $v_s = H/T$ rows per second to $H/T + v$. The time needed to store H rows changes from T to $H/(H/T + v)$, i.e., showing "compression" effect. The compression ratio is

$$\frac{H/(H/T + v)}{T} = \frac{H}{H + vT} = \frac{1}{1 + v/(H/T)} = \frac{1}{1 + v/v_s}$$

where $v_s = H/T$.

Part d (6 points).

As discussed in part.(c), when

$$v = -v_s = -H/T$$

the image appears to be constant in the y-direction.

Part e (6 points).

Suppose the unterlying time-varying image is $E(x, y, t)$. If all rows in an image are scanned simultaneously, then a particular scanned frame corresponds to $E(x, y, t)$ for some fixed t .

If instead the rows are sampled one after another, then the sampled image data can be consider to come from an apparent time-varying image $E'(x, y, t)$ as followed:

$$E^{'}(x, y, t) = E(x, y, t - ky)$$

where $k = T/H$. (T: frame time, and H: the image height.)
Thus we prove that

$$E^{'}(x, y, t) = E(x, y, t + ax + by)$$

where

$$\boxed{a = 0, \quad b = -T/H = -1/v_s}$$

This is assuming y increases upward, and rows are scanned top to bottom (flip the sign of k (b) if scanning is bottom to top). A particular scanned frame then corresponds to $E'(x, y, t)$ for some fixed t .

Taking partial derivatives, we find:

$$E^{'}_x = E_x, \quad E^{'}_y = E_y - kE_t, \quad \text{and} \quad E^{'}_t = E_t$$

From the image sequence we can estimate $E^{'}_x$, $E^{'}_y$, and $E^{'}_t$, but to compute image motion, we need E_x, E_y, and E_t . We see that

$$E_x = E^{'}_x, \quad E_y = E^{'}_y + kE^{'}_t, \quad \text{and} \quad E_t = E^{'}_t$$

So in our standard solution of the constant flow velocity problems we simply use

$$E^{'}_y + kE^{'}_t = E^{'}_y + (T/H)E^{'}_t$$

instead of E_y.
Note: If E_x and E_y are expressed as brightness change per pixel spacing (rather than brightness change per unit length), and E_t as brightness change per frame time (rather than brightness change per unit time interval), then the above expression becomes

$$E_y = E^{'}_y + (1/n)E^{'}_t$$

where n is the number of rows in the image.