

6.972 Game Theory and Equilibrium Analysis

Homework 6

April 23, 2005

Please choose and submit any 4 of the below problems for Homework 6.

DUE: In lecture 23

Problem 6.1 (War of Attrition)

Consider an incomplete-information version of the war of attrition of two players, in which the values are privately known. Player i chooses the time to concede $t_i \in [0, +\infty)$ according to his/her strategy $s_i(v_i) = t_i$, where v_i is player i 's value of the object, which takes value in $[0, +\infty)$ with cumulative distribution function F and probability density function f . The valuations are independent between players. Both players choose simultaneously. The payoffs are

$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ v_i/2 - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j \end{cases} \quad (1)$$

1. Characterize a symmetric Bayesian Nash equilibrium. (Assume that first order conditions are sufficient.)
2. Let $F(v) = 1 - \exp(-v)$ for $v \geq 0$. Derive the symmetric Bayesian Nash equilibrium explicitly.
3. Relate the war of attrition with incomplete information to the auction forms that we have seen. Assume that there exists a symmetric equilibrium $s(v)$, increasing in v , such that the expected payment of a bidder with value 0 is 0. Use the revenue equivalence theorem to derive a symmetric Bayesian Nash equilibrium.

Problem 4.2 (Reserve Prices)

In many instances, sellers reserve the right to not sell the object if the price determined in the auction is lower than some threshold amount, say $r > 0$. Such a price is called *reserve price*. In this problem, we would like to examine the effects of such a reserve price on the expected revenue accruing to the seller.

- (a) (Reserve Prices in Second-Price Auction) What is the expected payment of a bidder with value x in a second-price auction with reservation price r ?
- (b) (Reserve Prices in First-Price Auction) What is the expected payment of a bidder with value x in a first-price auction with reservation price r ?
- (c) (Revenue Effects of Reserve Prices) What is the optimal, or revenue maximizing, reserve price from the perspective of the seller?

Problem 4.3 (Mechanism Design - Grove's Mechanism)

Consider N agents with possible types $(\theta_1, \dots, \theta_N)$, where $\theta_i \in \Theta_i, \forall i$. These agents must make a collective choice from some set Y (allocation and transfer), e.g., $y = (x, t)$, with utility $u_i(y, \theta_i)$. For this problem, let's consider the quasi-linear utilities, s.t.,

$$u_i(y, \theta) = v_i(x, \theta_i) + t_i, \quad i = 1, \dots, N \quad (2)$$

We are interested in efficient allocations, s.t.,

$$g(\theta) = (x^*(\theta), t^*(\theta)) \quad (3)$$

We say that $g(\cdot)$ is efficient if the resulting $x^*(\theta)$ maximizes $\sum_{i=1}^N v_i(x, \theta_i)$, i.e.,

$$\sum_{i=1}^N v_i(x^*(\theta), \theta_i) \geq \sum_{j=1}^N v_j(x, \theta_j) \quad \forall x \in X \quad (4)$$

Show that "efficient allocation" is truthfully implementable in dominant strategies if the payments take the form

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(x^*(\theta), \theta_j) \right] + h_i(\theta_{-i}) \quad (5)$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i} .

Problem 4.4 (Communication Network: Shortest Path Problem)

Consider a communication network modelled by a directed graph G . There is a single source-destination pair (a, b) . The links are the strategic agents. The *private type* of link l is $c_l \geq 0$, which is the agent's cost of sending a single message (unit flow) along this link. Design a strategy proof mechanism that finds the shortest (lowest cost) path from a to b . Explicitly compute the payment involved. [You may find the following notation useful in writing the payment. Denote $d_{G|c_l=x}$: as the cost of the lowest cost path through G when c_l is set to be x]

Problem 4.5 (Strategy Proof Mechanism Design)

Consider a wireless system consisting of a base station transmitting to two radios. Let the total available power at the base station be P and suppose that the base station's goal is to allocate the power to the radios to maximize the sum of their utilities. Let the data rate to radio i when it is allocated power P_i be

$$R_i = \frac{1}{2} \log\left(1 + \frac{P_i}{\sigma_i^2}\right) \quad (6)$$

where σ_i^2 is the thermal noise power at the radio which cannot be controlled by the radio. Suppose that the utility of rate R_i to player i is given by $\omega_i R_i$. Clearly, if the base station knows $\{\omega_i\}$, then it can achieve its objective. Design a strategy-proof mechanism (where truth-telling is the dominant strategy for the radios) for the base station that is efficient. By efficiency, we mean that $\sum_i \omega_i R_i$ is maximized subject to the power constraint $\sum_i P_i \leq P$ and $P_i \geq 0$ for $i = 1, 2$. Explicitly compute the payments involved.

Problem 4.6 (Direct Revelation Mechanism)

Consider the direct revelation mechanism $\Gamma = (\Theta, g)$ where $g : \Theta \mapsto Y = X \times T$ with $g(\theta) = (x(\theta), t(\theta))$ and consider the quasilinear utility environment,

$$u_i(y, \theta) = v_i(x, \theta_i) + t_i, \quad i = 1, \dots, N.$$

Assume that $X = \mathfrak{R}$, the $v_i(\cdot, \theta_i)$ functions are twice continuously differentiable with $\partial^2 v_i(x, \theta_i)/\partial x^2 < 0$ and $\partial^2 v_i(x, \theta_i)/\partial x \partial \theta_i \neq 0$ at all (x, θ_i) , and each θ_i is drawn from an interval $[\underline{\theta}_i, \bar{\theta}_i] \subset \mathfrak{R}$.

a. Show that an efficient allocation is truthfully implementable in dominant strategies only if $t_i(\cdot)$ satisfies Eq. (5) [i.e., Groves mechanism is the only mechanism that implements efficient allocations and are strategyproof.]

b. (Budget balance) We say that the social choice function $g(\cdot) = (x(\cdot), t(\cdot))$ satisfies the budget balance condition if the payments satisfy $[\sum_i t_i(\theta) = 0, \quad \forall \theta]$. Let $N = 2$. Show that there is no mechanism that is strategyproof such that the corresponding social choice function g satisfies efficiency and budget balance. (or another term for both of these conditions "g is ex-post efficient".)

Problem 4.7 (Expected externality mechanism, or d'AGVA mechanism, due to d'Aspremont, Gerard-Varet, and Arrow)

Assume that the players' types are statistically independent. Let $x^*(\cdot)$ be an efficient allocation and assume that the payments take the form $t_i(\theta) = E_{\hat{\theta}} \left[\sum_{j \neq i} v_j(x^*(\theta_i, \hat{\theta}_{-i}), \hat{\theta}_j) \right] + h_i(\theta_{-i})$

a. Show that $(x^*(\cdot), t(\cdot))$ is truthfully implementable in Bayesian Nash equilibrium (or Bayesian incentive compatible).

b. Further assume that $h_i(\theta_{-i}) = -\frac{1}{N-1} \sum_{j \neq i} \xi_j(\theta_j)$, where

$$\xi_i(\theta_i) = E_{\hat{\theta}} \left[\sum_{j \neq i} v_j(x^*(\theta_i, \hat{\theta}_{-i}), \hat{\theta}_j) \right] \quad (7)$$

Show that $(x^*(\cdot), t(\cdot))$ satisfies budget balance condition.