| **6.972: Game Theory** | March 1, 2005 |
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# Lecture 8: Potential Games

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## 1 Introduction

In this lecture we visit the concepts of Potential Games, their equivalence to congestion games and an application to network design.

We begin with an example.

## 2 Example 1: Cournot Competition

- We have $n$ firms: $1, 2, ..., n$
- Firm $i$ chooses a quantity $q_i \in (0, \infty)$ and has a cost function $C_i(q_i) = cq_i$ for $c \in R_+$.
- Total Quantity produced is $Q = \Sigma q_i$.
- Inverse Demand (i.e., price) denoted as $P(Q)$.
- Payoff for player $i$: $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$.

We define $\Phi(q_1, \cdots, q_n) = q_1 \cdots q_n(P(Q) - c)$.

Note that $\forall i$,

$$\begin{aligned} u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) &> 0 \\ \Leftrightarrow \Phi(q_i, q_{-i}) - \Phi(q_i', q_{-i}) &> 0, \quad \forall q_{-i} \in \mathbb{R}_+^{n-1}, \forall q_i, q_i' \in \mathbb{R}_+ \end{aligned}$$

When $\Phi$ satisfies the above, it is called an *ordinal potential function* for the game.

## 3 Example 2: Cournot Competition Again

- We now have arbitrary differentiable costs i.e., $c_i(q_i)$.
- Price $P(Q) = a - bQ, \quad a, b > 0$.

Define the function

$$\Phi^*(q_1, \cdots, q_n) = a \sum_{i=1}^n q_i - b \sum_{i=1}^n q_i^2 - b \sum_{1 \le i < l \le n} q_i q_l - c \sum_{i=1}^n c_i(q_i).$$

Verify that, $\forall i, \forall q_{-i} \in \mathbb{R}_+^{n-1}, \forall q_i, q_i' \in \mathbb{R}_+$,

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi^*(q_i, q_{-i}) - \Phi^*(q_i, q_{-i}')$$

When $\Phi^*$ satisfies the condition above, it is called a potential function.

Note that if $q^*$ is a pure strategy Nash Equilibrium, i.e., $\forall i$,

$$\begin{aligned} & u_i(q_i^*, q_{-i}^*) \ge u_i(q_i, q_{-i}^*) \quad \forall q_i \\ \Rightarrow \quad & \Phi(q_i^*, q_{-i}^*) \ge \Phi(q_i, q_{-i}^*). \end{aligned}$$

More formally, given a game $\Gamma = \langle N, (S_i), (u_i) \rangle$, with $u_i : S \to R$,

**Definition 1 (Ordinal Potential)** *A function $\Phi : S \to \mathbb{R}$ is called an* ordinal potential function *for the game $\Gamma$ if $\forall i, \forall s_{-i} \in S_{-i}$,*

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0 \Leftrightarrow \Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0, \quad \forall x, z \in S_i.$$

*When $\Phi$ exists, the game $\Gamma$ is called an* ordinal potential game.

**Definition 2 (Exact Potential)** *A function $\Phi : S \to \mathbb{R}$ is called a* potential function *for the game $\Gamma$ if $\forall i, \forall s_{-i} \in S_{-i}$,*

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}), \quad \forall x, z \in S_i.$$

*When $\Phi$ exists, the game $\Gamma$ is called a* potential game.

Note, a potential funtion assigns one real value for every $s \in S$. Thus, when we represent the game payoffs with a matrix, we can also represent the potential function as a matrix, each entry corresponding to the vector of strategies from the payoff matrix. Here is an example.

| | |
|---|---|
| 1 , 1 | 9 , 0 |
| 0 , 9 | 6 , 6 |

Potential ==

| | |
|---|---|
| 4 | 3 |
| 3 | 0 |

Figure 1: Payoffs and potential function of a two-player potential game.

In the game shown in figure 1, verify that the matrix on the right is a potential function for the game with payoff matrix on the left.

Note that the maximum potential value corresponds exactly to the unique pure strategy NE of the game.

**Lemma 1** *Let $\Phi$ be an ordinal potential function for the game $\Gamma = \langle N, (S_i), (u_i) \rangle$. Then the set of pure strategy Nash Equilibria of $\Gamma$ is the same as the set of pure strategy Nash Equilibria of the game $\Gamma' = \langle N, (S_i), (\Phi) \rangle$, in which the payoff functions are all replaced by the ordinal potential function.*
*Equivalently, $s^*$ is a pure strategy Nash Equilibrium for $\Gamma$ iff*

$$\Phi(s_i^*, s_{-i}^*) \geq \Phi(s_i, s_{-i}^*) \quad \forall i, \forall s_i \in S_i.$$

*Proof:* Note that if $\Phi$ is an ordinal potential, then

$$\begin{aligned} u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) &= 0 \quad \text{iff} \\ \Phi(s_i^*, s_{-i}^*) - \Phi(s_i, s_{-i}^*) &= 0. \end{aligned}$$

Then $s^*$ is a pure strategy Nash Equilibrium

$$\begin{aligned} \Leftrightarrow u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) &\geq 0 \quad \forall i, \forall s_i \in S_i \\ \Leftrightarrow \Phi(s_i^*, s_{-i}^*) - \Phi(s_i, s_{-i}^*) &\geq 0. \end{aligned}$$

□

**Corollary 1** *If $\Phi$ is an ordinal potential function for a game and has a maximum value at $s^* \in S$, then $s^*$ is a pure strategy Nash Equilibrium for this game.*

**Remark:** Every strategy vector $s$ which maximizes the ordinal potential function will be a pure strategy equilibrium for the game, however there may be other pure strategy equilibria, which are local maxima of the ordinal potential. Thus, the set of global maxima of the ordinal potential may be used as a refinement concept over the pure strategy Nash Equilibria.

**Corollary 2** *Every finite ordinal potential game has a pure strategy Nash Equilibrium.*

How do we know which game has an ordinal potential?

In the following section, we characterize ordinal potential games. Note that unlike potential games, ordinal potential games are not easily characterized and the problem of finding a better characterization is still open.
(**Possible project topic:** Characterize the finite two-player ordinal potential games).

## 4 Characterization of Ordinal Potential Games

**Definition 3** *A* path *in a strategy space $S$ is a sequence of strategy vectors $(s^0, s^1, \cdots)$ such that every two consecutive strategies differ in one coordinate (one player's action).*
*An* improvement path *us a path $(s^0, s^1, \cdots)$ such that, $u_{i_k}(s^k) < u_{i_k}(s^{k+1})$ where $s^k$ and $s^{k+1}$ differ in the $i_k^{th}$ coordinate. In other words, the payoff improves for the player who changes his strategy.*

This can be thought of as an example of *myopic learning.*

**Proposition 1** *In every finite ordinal potential game, every improvement path is finite.*

*Proof:* Suppose $(s^0, s^1, \cdots)$ is an improvement path. Therefore we have,

$$\Phi(s^0) < \Phi(s^1) < \cdots,$$

where $\Phi$ is the ordinal potential. Since the game is finite, namely it has a finite strategy space, then the potential function takes on finitely many values and the above sequence must end in finitely many steps. □

The converse is not necessarily true. Consider the following example.

Every improvement path in this game is finite but there does not exist an ordinal potential.

**Proposition 2** *Let $\Gamma$ be a finite game in which every improvement path is finite, $u_i(x, s_{-i}) \neq u_i(z, s_{-i}) \forall x \neq z \in S_i, \forall s_{-i} \in S_{-i}$. Then $\Gamma$ has an ordinal potential.*

| | |
|---|---|
| 1,0 | 2,0 |
| 2,0 | 0,1 |

Figure 2: Example of game, in which all improvement paths are finite, but which does not have an ordinal potential.

*Proof:* Define the relation '>' as follows: $s > s'$ for $s, s' \in S$, if there exists a finite improvement path from $s'$ to $s$. Note that by the assumption of finite improvement paths, '>' is a transitive relation. Next, call a subset of the strategy space $Z \subseteq S$ *represented* if there exists a function $Q : Z \to \mathbb{R}$ such that $s > s' \to Q(s) > Q(s')$.

Suppose $Z$ is the maximal represented subset in $S$. We will show that $Z = S$. By way of contradiction, suppose there is a strategy $s \notin Z, s \in S$. Then we have the following possibilities for $s$.

- If $s > z$ for all $z \in Z$ then $Z \cup \{x\}$ is represented, by assigning $Q(s) = 1 + \max_{z \in Z} Q(z)$.
- If $z > s$ for all $z \in Z$ then $Z \cup \{x\}$ is represented, by assigning $Q(s) = min_{z \in Z}(Q(z)) - 1$.
- Otherwise define $Q(s) = \frac{a+b}{2}$ where $a = \max\{Q(z)) \mid z \in Z, s > z\}$ and $b = \min\{Q(y)) \mid y \in Z, y > s\}$.

In all three cases, we reach a contradiction with the maximality of $Z$, hence the whole strategy space $S$ is represented and the function $Q$ is an ordinal potential function for the game. □

## 5 Characterization of Exact Potential Games

For a finite path $\gamma = (s^0, s^1, \cdots, s^K)$ denote its overall change in potential by $\Delta(\gamma) = \sum_{k=1}^{K}[u_{i_k}(s^k) - u_{i_k}(s^{k-1})]$ where $i_k$ is the unique deviator from $s^{k-1}$ to $s^k$.
Call $\gamma$ a cycle if the $s^0 = s^K$ and the number of distinct vertices is finite.
Call $\gamma$ a simple cycle if $s^l \neq s^t$ for all $l \neq t$ other than $\{0, K\}$. The length of a simple cycle is the number of distinct vertices in the cycle.

**Theorem 1** *Given a game in strategic form $\Gamma$, the following claims are equivalent:*

1. *$\Gamma$ is a potential game.*
2. *$\Delta(\gamma) = 0$ for every cycle $\gamma$.*
3. *$\Delta(\gamma) = 0$ for every simple cycle $\gamma$.*
4. *$\Delta(\gamma) = 0$ for every simple cycle $\gamma$ of length 4.*

For a proof, see the paper "Potential Games" by Monderer and Shapley on our reading list.

**Corollary 3** *Consider the cycle of moves in figure 3. $\Gamma$ is a potential game iff for all $i, j \in N$, $a \in S_{-\{i,j\}}$ and $s_i, z_i \in S_i, s_j, z_j \in S_j$,*

$$u_j(B) - u_j(A) + u_i(C) - u_i(B) + u_j(D) - u_j(C) + u_i(A) - u_i(D) = 0.$$

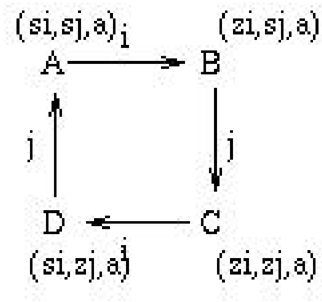


Figure 3: A cycle of length 4, where only players $i$ and $j$ deviate.

## 6 Congestion Games

Congestion games often arise in practice when users need to share resources in order to complete certain tasks, for example drivers who have different sources and destinations and need to share the roads. Every driver seeks a shortest or minimal cost path, and the cost of each road segment is adversely affected by the number of other drivers who use it.

Formally, we define a Congestion Model: $C\langle N, M, (S_i)_{i\in N}, (c^j)_{j\in M}\rangle$ where

- $N = \{1, 2, \cdots, n\}$ is the set of players;
- $M = \{1, 2, \cdots, m\}$ is the set of resources;

•$S_i$ consists of sets of resources (e.g., paths) that player $i$ can take. • $c^j(k)$ is the cost (negative payoff) to each user who uses resource $j$ if $k$ users are using it.

Based on this congestion model, we can define the congestion game $\langle N, (S_i), (u_i)\rangle$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j\in S_i} c^j(k_j),$$

where $k_j$ is the number of users of road $j$ under strategies $s$.

**Theorem 2** *Every congestion game is a potential game.*

Verify that

$$\Phi(s) = \sum_{j\in\cup s_i} (\sum_{k=1}^{k^j} c^j(k))$$

is a potential function for the congestion game. Another way of proof is from the characterization of potential games theorem.

The more remarkable result is that all finite potential games fall in the seemingly narrower class of congestion games.

**Theorem 3** *Every finite potential game is isomorphic to a congestion game.*

# 7 Application to Network Design

We define a network design game as follows. We are given a graph $G = (V, E)$ in which each edge $e$ has cost $c_e > 0$. We have a set $N$ of players. Each player has a set of terminal nodes $T_i \subseteq V$ that he wants to connect and his strategy space consists of subsets of edges connecting these nodes. The players have to pay for the edges they use, and they share the cost whenever more than one player uses an edge.
*Assumption:* $C_e$ is shared equally among all players that use $e$.
Thus, the cost of player $i$ is given by $C_i(s) = \sum_{e \in s_i} \frac{c_e}{x_e}$, where $x_e$ is the number of players using edge $e$.

The goal of each player is to connect his set of terminal nodes as cheaply as possible.

Consider the following example of a network with two parallel links.

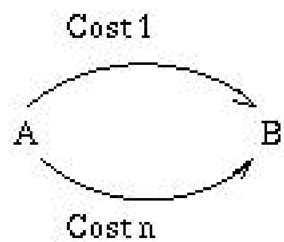


Figure 4: Network Congestion Game

One Nash Equilibrium in this case is when everyone uses the bottom link. Thus, cost of the worst Nash Equilibrium is $n$ times the minimum cost. This motivates us to look for the best Nash Equilibrium and study how close it is to the socially optimal solution (which may not be a Nash Equilibrium).

Define

$$\text{Price of stability} = \frac{\text{Cost of best Nash Equilibrium}}{\text{Cost of optimal solution}}.$$

**Theorem 4** *The price of stability in a network design game is at most* $H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \Theta(log(n))$.

The proof is based on a potential function. So far, potential functions have only been used to prove existence of Nash equilibria and this seems to be the first result in which they are used to bound the cost of Nash equilibria. This network design game falls in the class of congestion games and utilizes the same potential function we defined earlier in the section of Congestion games.

You can find the details in “The price of stability for Network design with fair cost Allocation” by Anshelerich et. al.