6.972: Game Theory April 21, 2005

Lecture 19

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1 Agenda

Auctions

- First price and second price
- Revenue equivalence principle

References:

- Auction Theory by Vijay Krishna
- Putting Auction Theory to Work by Milgrom

2 Auctions

We don't know other people's valuation (information is incomplete) and we assume a single object for sale. Each bidder knows his own valuation at the time of bidding and has some expectation about other people. This is called "private-value auctions" as opposed to "interdependent value auctions", where values of other people may affect my value.

- Auction formats: Bids are submitted simultaneously (sealed bid)
- Participant who submits the highest bid gets the object (standard auction)

We are going to be studying the *symmetric model*. More specifically we'll focus on the two models: *first price auctions* and *second price auctions*. Their bidding and allocation methods are the same, but the payoff calculations are different.

- First Price Auction: Winner pays the price she bids
- Second Price Auction: Winner pays the highest of the remaining bids

Auction format specifies a Bayesian game of incomplete information and we are interested in the equilibria states. (Incomplete information games: mapping from types to bids)

⇒ Each auction format determines a game of incomplete information among the bidders and our goal is to determine BNE and compare certain properties.

3 Symmetric Model

- N buyers bidding for a single object
- Bidder i assigns a value of X_i to the object
- Each X_i is iid on $[0, w]$ with cumulative distribution function F and density function $f = F'$
- Bidder i knows the realization x_i of X_i and that other bidder's values are independently distributed according to f.

$\Rightarrow$ Defines a Bayesian game (Type: Valuation)

A pure strategy for a bidder: $\beta_i : [0, w] \rightarrow R^+[0, \infty)$

Goal:

1. Find the symmetric equilibrium strategies in a first and second price auction.
2. From the point of view of the seller, which of the two auction formats yields a higher revenue?

Next week we'll discuss more of the mechanism design point of view. Here let's first see how we derive symmetric equilibrium strategies.

4 Second Price Auction

Second price is the simpler one.

- Each bidder submits a sealed bid b_i.
- Given these bids, the payoffs are:

$$\prod_i = \begin{cases} x_i - \max_{j \neq i} b_j & for \quad b_i > \max_{j \neq i} b_j \\ 0 & for \quad b_i < \max_{j \neq i} b_j \end{cases}$$

Proposition: In a second price sealed bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(x) = x$
Proof: Consider bidder 1, let $p_i = \max_{j \neq i} b_j$

Refer to Figure 1.

$\Rightarrow$ Symmetric NE

* Private values important (valuations known at the time of bidding)

Question: How much does each bidder expect to pay at this equilibrium?

- Fix bidder 1

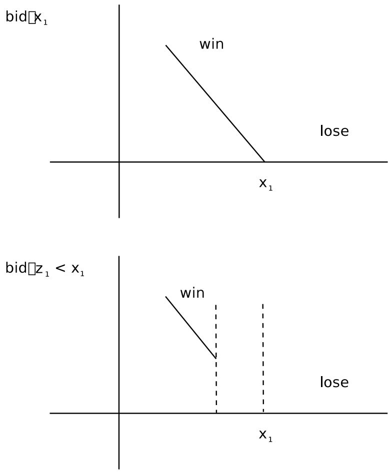

Figure 1: Different Bids

- Define the rv $Y_1 = \max\{X_2, \cdots, X_n\}$ whose cdf is $G(y) = Pr(Y_1 \leq y)$

$$G(y) = Pr(\max\{X_2, \cdots, X_n\} \leq y) = F(y)^{N-1} \qquad \text{by iid}$$

Expected payment by a bidder with value x_i

$$\begin{aligned} m^{II}(x) &= Pr(win) \cdot E[2nd\ highest\ bid|win] \\ &= G(x) \cdot E[Y_1|Y_1 \leq x] \end{aligned}$$

where $Pr(win) = G(x)$.

5 First Price Auction

- Each bidder submits a sealed bid b_i
- Given these bids $b_1, \cdots, b_n$ the payoffs are:

$$\prod_i = \begin{cases} x_i - b_i & for \quad b_i > \max_{j \neq i} b_j \\ 0 & for \quad b_i < \max_{j \neq i} b_j \end{cases}$$

So what happens to the quilibrium behavior? Given increasing: monotonic condtions. Suppose that bidders follow the symmetric increasing and differentiable equilibrium strategy $\beta^I = \beta$. People will tend to bid a little less than their own value to increase the payoff.

$*j \neq 1$ adopts β. Suppose that $X_1 = x$ and bidder 1 bids b. Find the payoff of the first guy's payoff and optimize over $b \rightarrow$ determine the optimal b.

5.1 Observation

- $\beta(0) = 0$
- Write bidder 1's expected payoff given other people's strategies and his type is x.
- Bidder 1 wins the auction if $b > \max_{i \neq 1} \beta(X_i)$. Since $\beta(\cdot)$ is increasing $\max_{i\neq 1} \beta(X_i) = \beta(\max_{i\neq 1} X_i) = \beta(Y_1)$

  Bidder 1 wins the auction whenever $\beta(Y_1) < b$ or equivalently $Y_1 < \beta^{-1}(b)$

His expected payoff:

$$\max_{b \geq 0} \ G(\beta^{-1}(b))(x-b)$$

How do we find this optimal b as a function of x?

Check first order conditions:

Note $f(f^{-1}(x)) = x$ so $f^2(f^{-1}(x))(f^{-1}(x) = 1$. Therefore,

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0$$

at a symmetric equilibrium $\beta(x) = b$ and $b = 0$ is not an optimal thing to do.

$\Rightarrow G(x)\beta'(x) + g(x)\beta(x) = xg(x)$ or equivalently $\frac{d}{dx}(G(x)\beta(x)) = xg(x)$

Since $\beta(0) = 0$ we have

$$\beta(x) = \frac{1}{G(x)} \int_0^x yg(y)dy \qquad \text{conditional expectation}$$

$$\beta(x) = E[Y_1 | Y_1 \leq x]$$

Let's try to understand the expected payment of each bidder:

5.2 Example

Assume values uniformly distributed over $[0,1]$

$$F(x) = x$$
$$G(x) = x^{N-1}$$
$$\beta^I(x) = \frac{N-1}{N}x$$

As an exercise try for exponential values $\lambda = 2$. Refer to Figure 2.

The revenue comparison:

$$\begin{aligned} m^I(x) &= Pr(win) \cdot amount\ bid \\ &= Pr(\max_{i \neq 1} \beta(X_i) < \beta(x)) \cdot amount\ bid \\ &= G(x)E[Y_1 | Y_1 <= x] \end{aligned}$$

- Ex-ante: expected payment of a particular bidder in either auction is: $E[m^A(x)] = \int_0^w m^A(x)f(x)dx = \int_0^w (\int_0^x yg(y)dy)f(x)dx$

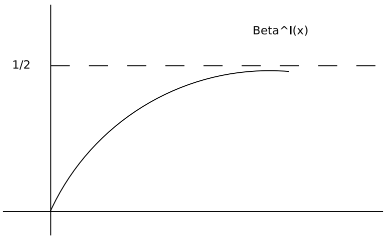

Figure 2: Exponential Bid

- Expected revenue of seller $E[R^A] = N \cdot E[m^A(X)]$

Remark: Other than the revenue being the same, the goods are allocated *efficiently* in the sense that the object goes to the person who values it most.

6 Revenue Equivalence Principle

- Extend the equality to a larger class of auctions.
- Auctions considered:
  1. Buyers submit bids
  2. "Standard auctions" (recall: highest bid gets the object)

6.1 Example: Third price auction

Propositon: Suppose that values are iid private (bidders are risk neutral, seek to maximize their payoffs). Then any symmetric and increasing equilibrium of any standard auction yields the same expected revenue.

Proof: Standard auction A
Symmetric increasing equilibrium: β
$m^A(x)$: expected payment by a bidder with value x
Assume $m^A(0) = 0$

Write down the payoff and optimize

$$\prod(z, x) = G(z)x - m^A(z)$$

but I bid $\beta(z)$ instead of $\beta(x)$ optimize over z and plugin x.

$$m^A(x) = G(x)E[Y_1|Y_1 \leq x]$$