Lecture 4: Extensive Games

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We have studied a lot about strategic game model which is a one-shot game in which each player chooses action once and for all simultaneously. In this lecture, we will study extensive game model where players engage in sequential decision making. The focus is on multi-stage games with observed actions where

1. All previous actions are observed.

2. Some players may move simultaneously at some stage k. If we including 'do nothing' in the strategy space, then all players move simultaneously.

Extensive form games can be conveniently represented by tree diagrams.

Example 1 (Entry Game) *There is a challenger who can choose to challenge the monopolist. The challenger is player one while the monopolist is player 2.*

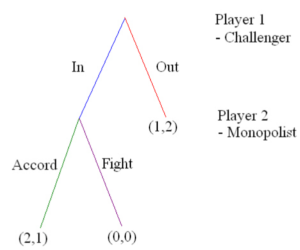


Figure 1: Entry Game

Example 2 (Investment in Doupoly) *There are two players in the market. Player one can choose to invest or not invest. After player one makes the choice, both players engage in the Cournot game.*

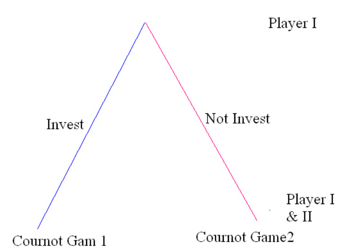


Figure 2: Investment in Duopoly

If player 1 invests, then they will engage in Cournot game 1 where $c_1 = 0$ *and* $c_2 = 2$*. Otherwise, they will engage in Cournot game 2 where* $c_1 = c_2 = 2$*. We can also assume that there is a fixed cost of* f *for player 1 to invest.*

1 Formal Game Model

In this section, we will present the formal model of an extensive game, which consists of four main elements.

a. A set of players, $\mathcal{I} = \{1, \ldots, I\}$

b. Histories: A set H of sequences which can be finite or infinite.

$h^0 = \emptyset$	initial history
$a^0 = (a_1^0, \ldots, a_I^0)$	stage 0 action profile
$h^1 = a^0$	history after stage 0
$\vdots$	$\vdots$
$h^{k+1} = (a^0, a^1, \ldots, a^k)$	history after stage k

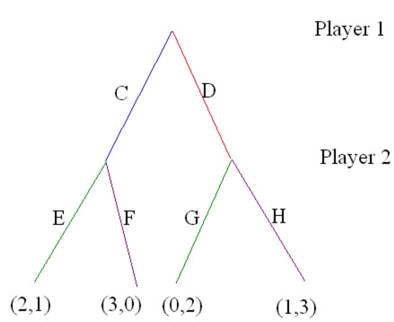
If the game has finite number $(K+1)$ of stages, then it is a finite horizon game. Let $H^k = \{h^k\}$ be the set of all possible stage k histories. Then H^{K+1} is the set of all possible *terminal histories*, and $H = \cup_{k=0}^{K+1} H^k$ is the set of all possible histories.

c. Pure Strategies for player i is defined as the contigent plan for every possible history h^k. Let $A_i(H^k) = \cup A_i(h^k)$ be the set of actions available to player i. Let $s_i^k : H^k \to A_i(H^k)$ such that $s_i(h^k) \in A_i(h^k)$. Then the pure strategy of player i is the set of sequences $s_i = \{s_i^k\}_{k=0}^K$. Observe that the path of strategy profile s will be $a^0 = s^0(h^0)$, $a^1 = s^1(a^0)$, $a^2 = s^2(a^0, a^1)$, and so on.

d. Preferences are defined on the outcome of the game H^{K+1}. Formally, it is the function $u_i : H^{K+1} \to \mathbb{R}$. As the strategy profile s determines the path $a^0, \ldots, a^k$ and hence h^{K+1}, we will denote $u_i(s)$ as the payoff to player i under strategy profile s.

It is important to point out that in extensive form game, a strategy specifies the action the player chooses for every history. Let's look at the following example.

Example 3 *Consider this game.*



The strategy of player 1 is the function $s_1^0 : h^0 \to A_1 = C, D$. The strategy of player 2 is the function $s_2^1 : H^1 = C, D \to A_2^1(H^1)$. There are four possible strategies for player two, namely EG, EH, FG and FH. If $s = (C, EG)$, then the outcome will be $\{C, E\}$. On the other hand, if the strategy is $s = (D, EG)$, the outcome will be $\{D, G\}$.

2 Solution Concept

In the strategic game, we study Nash equilibrium as one of the solution concepts. We will also define the Nash equilibrium for extensive form games.

Definition 4 *s^* is a pure strategy Nash equilibrium if for all $i \in \mathcal{I}$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all s_i.*

The Nash equilibrium is a reasonable prediction of the outcome in strategic game. However, this is not always the case in extensive form games. Let us go back to the entry game in example 1. The game has an equivalent strategic form representation as follows.

	Accord	Fight
In	2,1	0,0
Out	1,2	1,2

This game has two pure strategy Nash equilibria: $(In, Accord)$ and $(Out, Fight)$. However the second Nash equilibrium $(Out, Fight)$ is not reasonable because it is not optimal for player 2 if history $h_1 = In$ happens. Hence we will define a new equilibrium notion, which requires the strategy of each player to be optimal no only at the start of the game but also after every history.

Let h^k be a stage k history. We define $G(h^k)$ as the game ('subgame') from h^k on with

- Histories: $h^{K+1} = h^k, a^k, a^{k+1}, \ldots, a^K$.
- Strategies: $s_{i|h^k}$ is the restriction of s_i to histories in $G(h^k)$

Then a subgame perfect equilibrium is defined as follows.

Definition 5 *A strategy profile s is a subgame perfect equilibrium if for every h^k, the restriction $s_{|h^k}$ is a Nash equilibrium in $G(h^k)$.*

The definition of subgame perfect equilibrium provides a reasonable prediction for the outcome of an extensive form game. It is, however, hard to consider every subgame to check if a strategy is a subgame perfect equilibrium. The follow theorem provides a property of subgame perfect equilibrium which helps to verify if a strategy is a SPE.

Theorem 6 (One-stage deviation principle) *For finite horizon games, s^* is a subgame perfect equilibrium if and only if for all i, h^t and t, we have*

$$\begin{array}{ll} u_i(s_i^*, s_{-i}^*|h^t) \geq u_i(s_i, s_{-i}^*|h^t) & \\ \text{for all } s_i \text{ satisfying} & s_i(h^t) \neq s_i^*(h^t) \\ & s_{i|h^t}(h^{t+k}) = s^*_{i|h^t}(h^{t+k}) \forall k > 0, h^{t+k}. \end{array} \tag{1}$$

Proof. For one direction, if s^* is SPE, then the conditions are satisfied by definition.

For the other direction, suppose (1) holds. We have to show that s^* is SPE. To arrive at a contradiction, assume s^* is not SPE. Then there exists some subgame $G(h^t)$ such that $s^*_{|h^t}$ is not Nash equilibrium, i.e., there exists $\hat{s}_i$ such that $u_i(\hat{s}_i, s_{-i}^*|h^t) > u_i(s_i^*, s_{-i}^*)|h^t$.

Define $\bar{T} = \{\tau \geq t \mid \exists h^\tau with \hat{s}_{i|h^t}(h^\tau) \neq s^*_{i|h^t}(h^\tau)\}$. First $t \in \bar{T}$. By (1), $|\bar{T}| \geq 2$. We will reach a contradiction for $|\bar{T}| = 2$.

Suppose $|\bar{T}| = 2$. Then there exists $\hat{t} > t$ and $h^{\hat{t}}$ such that $\hat{s}_i(h^{\hat{t}}) \neq s_i^*(h^{\hat{t}})$. Construct $\tilde{s}_i$ such that $\tilde{s}_i(h^\tau) = \hat{s}_i(h^\tau)$ $\forall \tau < \hat{t}$ and $\tilde{s}_i(h^\tau) = s_i^*(h^\tau)$ $\forall \tau \geq \hat{t}$. By assumption, $u_i(\hat{s}_i, s_{-i}^*|h^t) > u_i(s_i^*, s_{-i}^*)|h^t$. By the one stage deviation equation, $u_i(s_i^*, s_{-i}^*|h^t) \geq u_i(\tilde{s}_i, s_{-i}^*)|h^t$. Hence $u_i(\hat{s}_i, s_{-i}^*|h^t) > u_i(\tilde{s}_i, s_{-i}^*)|h^t$.

Consider the subgame starting at $h^{\hat{t}}$. From the one stage deviation equation, $u_i(\tilde{s}_i, s_{-i}^*|h^t) = u_i(s_i^*, s_{-i}^*|h^t) \geq u_i(\hat{s}_i, s_{-i}^*)|h^t$. Then $u_i(\tilde{s}_i, s_{-i}^*|h^t) > u_i(\hat{s}_i, s_{-i}^*)|h^t$ because $\tilde{s}_i(h^\tau) = \hat{s}_i(h^\tau)$ for all $\tau < \hat{t}$. ■