

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
15.053 – Optimization Methods in Management Science (Spring 2007)

Problem Set 1

Due February 15th, 2007 4:30 pm in the 15.053 Box
You will need 100 points out of 124 to receive a grade of 5.

Problem 1: An optimized yarn (24 points, 4 points per part.)

The purpose of this problem is to practice formulating optimization problems and to start thinking about the different properties they contain.

Suppose we are given a piece of yarn that is 12m long.	
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Part A:

Our objective is to cut the yarn into four pieces and form a rectangle with maximum area. Formulate an optimization problem that when solved will give the lengths that maximize the area. Is your optimization problem a linear program?

Part B:

Solve the optimization problem from “part a” using results from single variable calculus. What is the optimum solution?

Part C:

Ollie the Owl, wise and curious, asks what is the largest area that can be formed if we cut the yarn into three pieces and form a triangle. Formulate an optimization problem whose solution will give you the optimal lengths. Be sure that your model eliminates spurious solutions such as 1, 1, 10.

Use Heron’s Formula: The area of a triangle with sides w , x and y is $\sqrt{s(s-w)(s-x)(s-y)}$, where $s = (w + x + y)/2$.

Part D:

Solve the optimization problem in part c and comment on the solution.

HINT 1: Suppose that $f(w, x, y) > 0$ for all w, x, y . A solution is maximum with respect to the objective function $\sqrt{f(w, x, y)}$ if and only if it is maximum with respect to the objective function $f(w, x, y)$.

HINT 2. Use results from single variable calculus to show that for any fixed value of y with $0 < y < 6$, the optimal solution will have $w = x$. Does your proof also show that $x = y$ in an optimal solution?

Part E:

Suppose you are asked to cut the yarn into four pieces to form a quadrilateral with maximum area. What do you think the optimal solution is? Formulate an optimization problem that when solved will give you the optimal solution.

Hint: Go to mathworld.wolfram.com/BrahmaguptasFormula.html and use Brahmagupta's Formula for your objective function, assuming that $\cos(.5(A+B)) = 0$

Part F: (Bonus 2 Points):

Solve the problem in part e, assuming that $\cos(.5(A+B)) = 0$. Did you get the solution you were expecting?

Problem 2: Beer Production (28 points, 4 points per part)

The goal of this problem is to introduce you to a production problem and to practice adding additional constraints to a model as new restrictions become known. The key to this problem is to define the proper decision variables.

One of the main uses of linear programming is to determine an optimal allocation of resources for a firm that produces multiple goods with limited resources. In this problem we develop a model for the Anteater-Bugs beer company. We start out with a basic core problem and then add additional constraints to further enhance the applicability of the model.

The Anteater-Bugs corporation has two main brands of beer: Bugwheezer and Bug-Lite. Each of these products contains two main ingredients Malts and Hops. According to recent media reports:

The Super Bowl represents an enormous commitment for Anteater-Bugs. Arthur, chief creative officer of Anteater, as well as it's mascot, said the brewer has been advertising on the game since 1976. "It's the most efficient way to reach the most adult consumers in one sitting," Arthur said Wednesday. And we also reach a lot of underage drinkers, which bothers us a lot ☺.

Hence the demand for each of these products increases after the Super Bowl. The following table provides the data for producing the products. (All of the data is in units per bottle). Note: Labor is expressed in hours.

Ingredient/Beer	Bugwheezer	Bug-Lite	Available
Malts	10	3	1000
Hops	20	12	2000
Labor	3	3	1500
Profit	\$6	\$4	

Part A:

Formulate a linear program for the Anteater-Bugs Corporation to determine the optimal post Super Bowl production quantities to maximize profits subject to the available resources. You do not need to solve the program.

Part B:

One piece of data the above model is missing is the demand for beer. Based on estimates from prescreening of ads, the demand for Bugwheezer is 3000 and the demand for Bug Lite is 2000. Every unit produced above these quantities will be unsold and hence not contribute to the profit. Show how to modify your linear program from “part a” to take this into account.

Part C:

Another aspect of production not taken into account is machine time. To produce a bottle of either beer requires one hour of machine time. There are 4000 hours of machine time available. Modify your linear program from “part b” to take this into account.

Part D:

A constraint is called *redundant* if when the constraint is removed from the linear program, the feasible region is unchanged; that is, no new feasible solution is created by elimination of the constraint. Are any of the constraints in part c redundant? If so which one(s)? If not, say so. In either case, justify your answer.

Part E:

Suppose that the demand for bottles of Bugwheezer and Bug Lite from part b are due to orders from retailers, and there is a penalty for not satisfying this demand. There is a penalty of \$1 per bottle of Bugwheezer and \$2 per bottle of Bug Lite for each bottle short of meeting the demand. *For example if 1800 bottles of Bug Lite are produced, then 200 units of demand are not met, and the corporation incurs a penalty of \$400 (200 unfilled * \$2 per bottle).* Modify your linear program from “part b” to take into account the penalty incurred if the demands of 3000 and 2000 are not met. (HINT: you will need to create additional variables.)

Part F:

One possibility of avoiding a production shortage is to pay labor overtime to increase the total number of labor hours. Suppose additional labor can be purchased for \$1 per hour up to 2000 hours of additional labor. Take your formulation from “part e” and modify it to include the option of using overtime labor.

Part G:

Suppose that Anteater-Bugs produces n brands of beer, labeled B_1, B_2, \dots, B_n . Brand B_j requires m_j units of malt, h_j units of hops, l_j hours of labor, has a demand of d_j , incurs a profit of p_j for each unit sold up to the number demanded, and a cost of c_j for each unit of demand not met. Formulate the abstracted version of the problem.

Problem 3: Conversions (12 points, 2 points per part.)

An important skill is to be able to convert certain types of optimization problems to linear programs. In this problem, we practice identifying such problems and the methods used to convert them to linear programs.

Optimization Problem 1:

Consider the following optimization problem:

$$\begin{aligned} \max \quad & \min\{3x + 4y, 3x, qz + 4x\} \\ \text{s.t.} \quad & \frac{x + y}{3x - 1} \leq 8 \\ & x, y \leq 0 \end{aligned} \tag{OPT1}$$

In the above program, q is some positive constant.

Part A:

Is OPT1 linear? If not explain which part(s) of it are not linear.

Part B:

Is the objective function in OPT1 a convex function? Is it a “friendly function”, where “friendly” is described in Lecture 2?

Part C:

If you answered no to “part a”, can OPT1 be converted to a linear program? If so, convert it to a linear program showing all steps. If not explain why not.

Optimization Problem 2:

Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & 3x + qy \\ \text{s.t.} & x + 2y \leq 30 \\ & y = |x| \\ & x \leq 0 \end{array} \quad (\text{OPT2})$$

In the above problem q is some positive constant.

Part D:

Is OPT2 linear? If not explain which part(s) of it is not linear.

Part E:

Can OPT2 be converted to a linear program? If so convert it to a linear program showing all steps. If not, explain why not.

Part F:

Under what conditions on q is the solution $x = -30, y = 30$ optimal? Under what conditions on q is the solution $x = 0, y = 0$ optimal? Are there any circumstances where neither one of these two solutions is optimal? You don't need to prove your answer.

Problem 4: Kevin Federline's Ice Cream Shop (20 points, 4 points per part.)

This problem is designed to teach you how to model multi period inventory models and how to convert some nonlinear programs to linear programs. As with Problem 2, the key is to define the proper decision variables.

After his split with Britney, Keven Federline has realized he needs a new source of income to support his "partying". After a well-publicized shortage of success at dancing, singing, acting, producing, insect collecting, teaching, grocery bagging, surreal lifeing, and most recently burger flipping he has decided to start his own ice cream store. The store will sell a single flavor: pumpkin. He will run the shop only in September, October, and November. He needs the rest of the year for vacation. The demand (in scoops per month) is 200, 250, and 600. Demand peaks around Thanksgiving. The cost to make a scoop is \$2 in September and October and increases to \$6 for November. This is due to a shortage of pumpkins after Halloween. No more than 400 scoops can be produced in a single month.

Part A:

Under current conditions, Kevin cannot store ice cream from one month to the next. Explain to Kevin (preferably using small words) why this problem is not feasible.

Part B:

Suppose now Kevin invests in a freezer so that pumpkin ice cream produced in month i can be stored and sold during month $i+1$. (In fact, one can store ice cream for as many months as desired.) For every month pumpkin ice cream is stored, a \$3.50 storage cost is incurred per scoop of ice cream. With the new possibility of storing pumpkin ice cream, formulate the problem of meeting demand at minimum cost as a linear program.

Part C:

Kevin forgot to mention that the freezer has a limited capacity. However he threw out the freezer brochure and is too lazy to measure the capacity. Suppose now we add the restriction that a maximum of q units can be stored each month (e.g., the freezer can only hold q units). Add this new condition to the formulation from part b.

Part D:

What is the minimum freezer size q so that the problem in part c is feasible? Explain so that someone with Kevin's intelligence (or at least close to) could understand why this is the smallest possible value of q .

Part E:

Suppose Kevin is operating the ice cream shop for n months. The demand for ice cream for month i is d_i . The cost of producing a scoop of ice cream in month i is c_i . The storage cost for each month is s , and the freezer has a capacity of j scoops. Formulate the abstracted version of Kevin's Problem.

Problem 5: Candy Production Blending Problems (20 points, 6, 6, 8 Points)

Willy Wonka's Candy Company Produces three types of Candy

1. Wonka Bars
2. Bottle Caps
3. Giant Sweet Tarts

In order to produce the different type of candies, Willy can run three different production processes as described below. Each process involves blending different types of sugars in the Magical Factories Mixer.

Process 1:

Running Process 1 for one hour:

Costs: \$5

Requires: Two barrels of sugar type A and three barrels of sugar type B

Output: Two Wonka Bars and one packet of Bottle Caps

Process 2:

Running Process 2 for one hour:

Costs: \$4

Requires: One barrel of sugar type A and three barrels of sugar type B

Output: Three packets of Bottle Caps

Process 3:

Running Process 1 for one hour:

Costs: \$1

Requires: Two barrels of sugar type B and three packets of Bottle Caps

Output: Two Wonka Bars and one packet of Giant Sweet Tarts

Incidentally, the barrels are really tiny, and the candy is very sweet.

Each week we can purchase:

- 200 barrels of sugar type A at \$2 Per Barrel
- 300 barrels of sugar type B at \$3 Per Barrel

Assume that they can sell everything that they can produce.

- Wonka Bars are sold at \$9 per bar.
- Bottle Caps are sold at \$10 per packet.
- Giant Sweet Tarts are sold at \$24 per packet.

Assume that 100 hours of mixing time are available.

Part A: Formulate an LP whose solution will maximize Willy Wonka's Profits.

Part B: Assume that instead of having 200 barrels of sugar type A and 300 barrels of sugar type B available that you can order a total of 500 Barrels. Show how to modify your LP formulation in Part A to account for this revised problem.

Part C: Suppose that instead of selling the three candies separately, they can only be sold as part of a box consisting of one Wonka Bar, two packets of Bottle Caps, and one pack of Giant Sweet Tarts. Each Wonka Box sells for \$54. Modify your LP formulation in part A to model this new scenario.

Problem 6: Convex Functions (10 points, 2 points per part.)

In this problem we explore the properties of convex functions through a few short answer questions. In each case, assume that each function $f(x)$ is of a single variable.

Part A:

Is it possible for a function of one variable to be both convex and concave? If so give an example. If not, explain why not.

Part B:

Is it possible for a function to be neither convex nor concave? If so, draw such a function. If not, explain why not.

Part C:

Suppose that $f(x)$ and $g(x)$ are both convex, and let $h(x) = \min\{f(x), g(x)\}$. Is it necessarily true that $h(x)$ is convex? If yes, explain your answer. If no give a counter example.

Part D:

Suppose that $f(x)$ and $g(x)$ are both convex, and let $h(x) = f(x) + g(x)$. Is it necessarily true that $h(x)$ is convex? If yes, explain your answer. If no give a counter example.

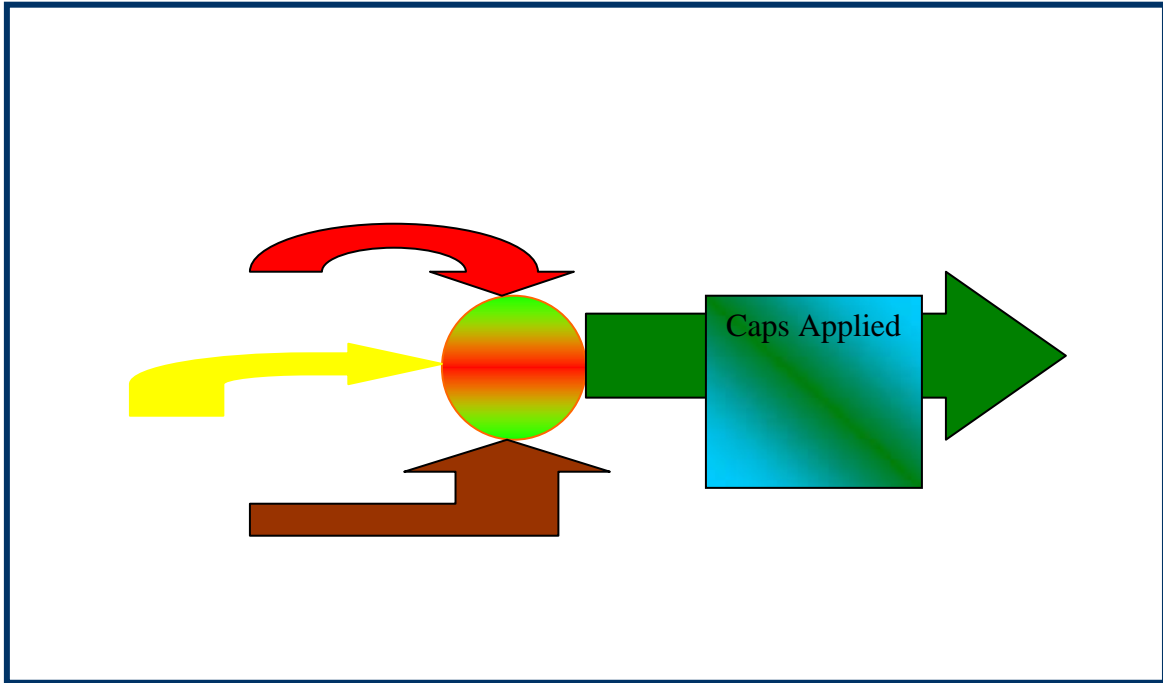
Part E:

Suppose that $f(x)$ and $g(x)$ are both convex, and let $h(x) = f(x) \times g(x)$. Is it necessarily true that $h(x)$ is convex? If yes, explain your answer. If no give a counter example.

Extra Credit Challenge Problem: Nooz Bottling Plant(10 Points, 5 Points Per Part)

This problem is intended to give you practice in trying to model continuous time situations as a linear program.

Nooz bottling plant produces three different types of soda (red, yellow, and brown),the bottles are produced on separate assembly lines and converge at a single point where Nooz applies the caps. This is shown below.



The profit for bottles are: \$3 for red, \$4 for yellow, and \$5 for brown. The production rate from the red, yellow, and brown bottles are 500, 600, and 400 bottles per hour. When the bottles arrive at the intersection there is a gate. The gate can only accept bottles from one assembly line at a given time. If the gate is pointing yellow then all yellow bottles that arrive are instantly capped and \$4 is earned for each bottle. When the gate is on yellow any red or brown bottles that arrive are dumped into the trash and no money is earned by them. The gate is on a 2.2 min cycle. During this 2.2 minute cycle the gate must be on each bottle type for at least 25 seconds and is followed by a 10 second changing period where the gate is turned off(all bottles that arrive during this period are lost).

Part A:

Formulate a linear program to determine the optimal amount of time in each cycle that the gate will be accepting bottles from each bottle type to maximize revenue per cycle.

Hint: The number of red bottles that will pass through the gate in a given cycle is 500* (seconds gate is red in cycle/ 1 hour). Also it will be much easier if you convert all units to seconds.

Part B:

Suppose now due to a labor union constraint that plant can cap a maximum of 510 bottles per hour. Incorporate this constraint into your linear program