3.044 Problem Set 1

Heat Conduction

Solutions

1. Thermal properties and optimal materials selection

   (a) The steady flux is given by Fourier's law: $q = -k\frac{dT}{dx}$. If we have a hot body and a cold body at certain temperatures and a certain distance apart, then $\frac{dT}{dx}$ is fixed, and we want to minimize k. Kevlar has by far the lowest k (and is a satisfactory answer to the problem), but as a fiber, it needs a matrix material, and a Kevlar-polycarbonate composite will have the lowest dense material thermal conductivity.

   (b) We want a long timescale, so that the heat bursts are damped by the heat shield. Since the timescale is $\frac{L^2}{\alpha}$, we want to minimize α, and the best material is again Kevlar, or a Kevlar-polycarbonate composite (though polycarbonate is close enough that if cost is an issue it might win).

   (c) Here, we want short timescale, so we maximize α (and exclude diamond), giving us silver as the optimal material.

   (d) All we need to do is maximize heat energy per unit weight per degree, and the heat capacity c_p measures just that. Unfortunately, the graph does not provide sufficient information, so we need to just choose among those listed, of which alumina at 804 has the highest value.

   Note that in these units water has a heat capacity of 4184 $\frac{\mathrm{J}}{\mathrm{kg \cdot K}}$, which is over five times better, so a hydrate or gel would fare much better than any of these.

   (e) We want to minimize ΔT for a given q. If we solve Fourier's 1st law for ΔT, we find it is equal to $\frac{qL}{k}$. With q and L fixed, minimizing ΔT means maximizing k, and the choice is diamond.

   (f) Here, we want to maximize the flux for an unsteady problem. If we look at the erf(c) solution, which is valid for the time of initial contact between the molten metal and the rotating wheel, we find that $T = T_i + (T_0 - T_i)\mathrm{erfc}\left(\frac{y}{2\sqrt{\alpha t}}\right)$, where y is the distance from the wheel's outer surface. If we evaluate the flux through the surface using $q = -k\frac{dT}{dy}$, this gives us $q = -k(T_0 - T_i)\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\alpha t}}$. The flux is proportional to $\frac{k}{\sqrt{\alpha}}$, which is equal to $\sqrt{k\rho c_\mathrm{p}}$. The non-diamond material with the largest value of this parameter is copper. (Perhaps as diamond films fall in price they too will be used in this application.)

Candidate materials (omitting those in the middle which aren't the best at anything):

Material	k, $\frac{\text{W}}{\text{m}\cdot\text{K}}$	ρc_p, $\frac{\text{J}}{\text{cm}^3\cdot\text{K}}$	α, $\frac{\text{cm}^2}{s}$	$\sqrt{k\rho c_p}$, $\frac{\text{W}\sqrt{\text{s}}}{m^2\cdot K}$
diamond	2320	1.8	12.8	6.49×10^4
silver	425	2.46	1.73	3.23×10^4
copper	346	3.4	1.02	3.43×10^4
gold	315.5	2.51	1.26	2.81×10^4
aluminum	243	2.43	1.0	2.43×10^4
magnesium	135	1.78	0.75	1.55×10^4
Nickel 270	86	4.1	0.21	1.88×10^4
graphite	63	1.60	0.39	1.00×10^4
alumina (Al_2O_3)	39	3.18	0.122	1.11×10^4
Carpenter HyMu	34.6	4.3	0.08	1.22×10^4
lime (CaO)	15.5	2.48	0.0623	6210
Haynes 188	10.4	3.6	0.0028	6120
Steatite	3	2.5	0.012	2740
polycarbonate	0.2	1.60	0.0012	570
Kevlar	0.04	0.49	0.008	140

2. Layered furnace wall and British units

   Set R_1 to the inner radius and T_1 to the temperature there, which equals the melt temperature of 2000°F. Set the radius of the graphite-brick interface to R_2, and the temperature there to T_2 (not given). Set the outer radius of the brick later to R_3, the outer brick temperature to T_3 (also not given), and the environment temperature to T_4 (70°F).

   (a) This is a sum of resistances problem, with the linear solution for the top and bottom given in class:

$$q_z = \frac{T_1 - T_4}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h}}$$

   The first material is graphite, L_1 is 1.5 ft and k_1 is 63 $\frac{\text{W}}{\text{m}\cdot\text{K}} \times 0.557 = 35.1 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}$; for brick, $L_2 = 4$ ft and $k_2 = 16 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}$; $h = 4 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\cdot{}^\circ\text{F}}$. So this gives us

$$q_z = \frac{1930^\circ\text{F}}{0.543 \frac{\text{hr}\cdot\text{ft}^2\cdot{}^\circ\text{F}}{\text{BTU}}} = 3556 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2}$$

   The area of the top and bottom are πR^2, or 314 ft^2 each, so twice this times the flux gives $2.23 \times 10^6 \frac{\text{BTU}}{\text{hr}}$.
   In the radial direction, the equivalent is written in terms of Q, the flux-area product:

$$Q = \frac{2\pi L(T_1 - T_4)}{\frac{1}{k_1}\ln\frac{R_2}{R_1} + \frac{1}{k_2}\ln\frac{R_3}{R_2} + \frac{1}{hR_3}}.$$

   With our parameters, this gives us

$$\frac{1.82 \times 10^5 \text{ft} \cdot{}^\circ \text{F}}{0.0388 \frac{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}{\text{BTU}}} = 4.69 \times 10^6 \frac{\text{BTU}}{\text{hr}}$$

   Adding the power through the sides to that through the top and bottom gives a total power of $6.92 \times 10^6 \frac{\text{BTU}}{\text{hr}}$. You could convert the units using 1 BTU=1055 J, and 1 hr=3600 s, so the total power is 2030 kW (but you didn't have to).

(b) By ignoring the corners, we treat them as perfect insulators. In a real furnace, they too would conduct heat away, increasing the loss of thermal energy. So our power number is an *underestimate.*

(c) We can just use the equation above with our known Q and a single layer at a time, *e.g.* for the graphite layer:

$$Q = \frac{2\pi L(T_1 - T_2)}{\frac{1}{k_1}\ln\frac{R_2}{R_1}}$$

Solve for T_2:

$$T_2 = T_1 - \frac{Q\left(\frac{1}{k_1}\ln\frac{R_2}{R_1}\right)}{2\pi L}$$

For our parameters and Q, we arrive at $T_2 = T_1 - 198°\text{F} = 1802°\text{F}$. Doing the same for the brick layer gives $T_3 = T_2 - 929°\text{F} = 873°\text{F}$. Finally, the analogue for the outer heat transfer coefficient is

$$T_4 = T_3 - \frac{Q}{2\pi L h R_3}$$

This gives $T_4 = T_3 - 803°\text{F} = 70°\text{F}$, which is the outside temperature, as it should be.

3. Joule heating of a titanium rod

(a) Start with the solution to the cylindrical heat conduction equation with uniform heat generation at steady state, from the handout "1-D Thermal Diffusion Equation and Solutions":

$$T = -\frac{\dot{q}r^2}{4k} + A\ln r + B$$

Since the temperature is finite at $r = 0$, we know $A = 0$. The boundary condition $r = R \Rightarrow T = T_s$ (surface temperature) gives us:

$$T = T_s + \frac{\dot{q}}{4k}(R^2 - r^2)$$

The maximum temperature is at $r = 0$, and the minimum at $r = R$; the difference is:

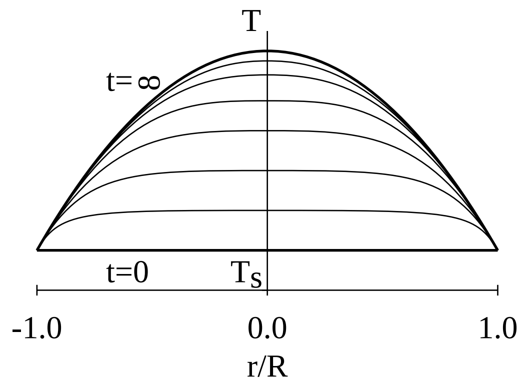
$$T_{max} - T_{min} = \frac{R^2\dot{q}}{4k}$$

For $R = 1.25 \times 10^{-3}\text{m}$, $\dot{q} = 5 \times 10^6 \frac{\text{W}}{\text{m}^3}$, and $k = 20 \frac{\text{W}}{\text{m}\cdot\text{K}}$, this gives

$$T_{max} - T_{min} = 0.098\text{K}$$

(b) The temperature difference is proportional to $\dot{q}$, so if that quadruples (because Joule heating goes as the current density squared), the temperature difference quadruples to about 0.4K.

(c) Here you had to estimate a sketch of something you've never seen before. You know the initial condition at $t = 0$ is uniform temperature at $T = T_s$, and the long-term steady-state distribution is given by part 3a. In between, it should heat up pretty uniformly in the middle, until it reaches that steady-state. So it will look something like:

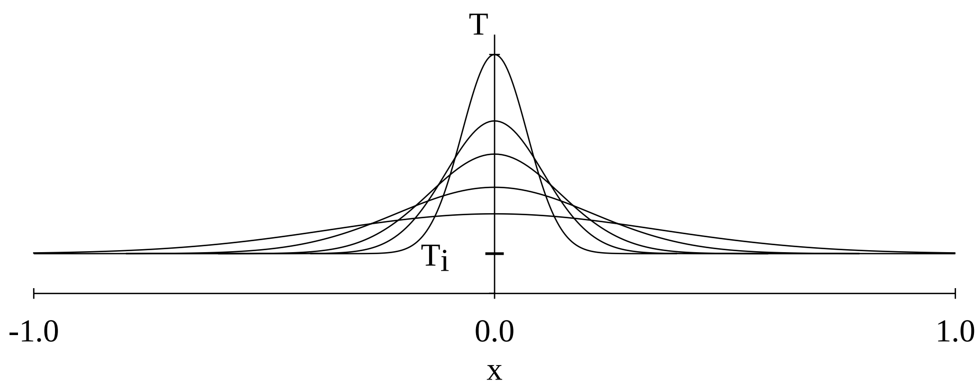


(d) This is just the steady-state criterion:

$$t_{SS} = \frac{R^2}{\alpha} = \frac{R^2 \rho c_p}{k} = 0.26\text{seconds}$$

4. Heat Transfer in Resistance Welding

(a) With a fixed amount of excess heat deposited in a thin layer at the junction, and that heat diffusing out along the lengths of the rods, this looks like the Gaussian:



(b) At very short time scales, we have no idea what the temperature distribution looks like. At moderate to long time scales, we can use the Gaussian solution, which is given by:

$$T = T_i + \frac{(T_0 - T_i)\delta}{\sqrt{\pi \alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right),$$

where T_i is the initial temperature of most of the piece, δ is the thickness of the layer at temperature T_0 (half the thickness in this two-sided problem), and α the thermal diffusitvity $k/\rho c_p$.

(c) We have the energy per unit cross-section area of weld, and we need something like $T_0 - T_i$ and δ, both of which are unknown. To get there, we can use the relationship between temperature change and volumetric enthalpy density change:

$$\Delta H = \rho c_p \Delta T \Rightarrow \Delta T = \frac{\Delta H}{\rho c_p}$$

Here when we divide enthalpy per unit area by ρc_p, we get something interesting:

$$\frac{3 \times 10^6 \frac{\text{J}}{\text{m}^2}}{2700 \frac{\text{kg}}{\text{m}^3} \cdot 917 \frac{\text{J}}{\text{kg}\cdot\text{K}}} = 1.21\text{K} \cdot \text{m}$$

These units K·m are exactly the units of $(T_0 - T_i)\delta$. In fact, just as in diffusion the area under the Gaussian represents the total solute content, which is fixed, the area under the thermal Gaussian represents the total heat content per unit cross-section area, which is fixed.

Because δ represents half of the thickness of the original heated region in this formulation, we need to use half of the energy for $(T_0 - T_i)\delta$, and can then look for T at $x = 0$:

$$T = 40°\text{C} + \frac{0.606\text{K}\cdot\text{m}}{\sqrt{\pi \cdot \frac{238\frac{\text{W}}{\text{m}\cdot\text{K}}}{2700\frac{\text{kg}}{\text{m}^3}\cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}} \cdot 1\text{second}}} \exp(-0) = 40°\text{C} + 35°\text{C} = 75°\text{C}.$$

At $t = 10$ seconds, the maximum temperature difference is a factor of $\sqrt{10}$ lower, or 11°C, making the maximum temperature 51°C.

(d) The maximum temperature is 35°C or 11°C above the temperature of the rest of the rod. To calculate the width where temperature difference is at least half that, we need only calculate where:

$$\exp\left(-\frac{x^2}{4Dt}\right) \geq \frac{1}{2}$$

$$x^2 = 4\alpha t \ln(2) \Rightarrow x = 2\sqrt{\frac{238\frac{\text{W}}{\text{m}\cdot\text{K}}}{2700\frac{\text{kg}}{\text{m}^3} \cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}} \cdot 1\text{second} \cdot \ln(2)} = 0.016\text{m}.$$

This is the distance along the rod from $x = 0$ where this is satisfied, so the full length of this region in both directions from the weld is twice this, or about 3.2 cm.

At $t = 10$ seconds, the full width will be a factor of $\sqrt{10}$ wider, or about 10 cm.