3.044 Problem Set 4

Solidification and Fluids

Due Wednesday March 30, 2005

Note: you should do *either* problem 3 *or* problem 4, *not* both (that's why the points add up to 128).

1. Heat conduction and diffusion in alloy casting (30 pts)

   In the die-casting of a relatively thick roughly plate-shaped metal alloy part, the liquid metal alloy cools and reaches a roughy uniform distribution at the melting point at time $t = 0$, then solidifies with a plane front from the sides. The rate of solidification is limited by two types of heat transfer: conduction through the mold and to the environment can be represented by $q_x = h(T_s - T_{env})$ (T_s is the outer surface metal temperature), and conduction through the already-solidified metal of thickness Y.

   (a) Sketch the temperature profile (T vs. x, where x is the distance from one side of the mold) across the solid metal shell and liquid metal interior for short times (small Bi) and long times (large Bi). (5)

   (b) Give a simple expression for the growth rate dY/dt which is valid for short times (small Bi). (4)

   (c) Derive a simple expression for the growth rate dY/dt which is valid for long times. (4)

   (d) Sketch the relationship between solidified shell thickness Y and time t, showing the transition from convection- to conduction-limited growth. (5)

   As the metal solidifies, the lower solubility of the alloying element (solute) results in its "rejection" into the liquid. At steady-state, the concentration of the solute in the solid is that of the liquid C_L, but in the liquid at the liquid-solid interface, the concentration is much higher, say $5C_L$.

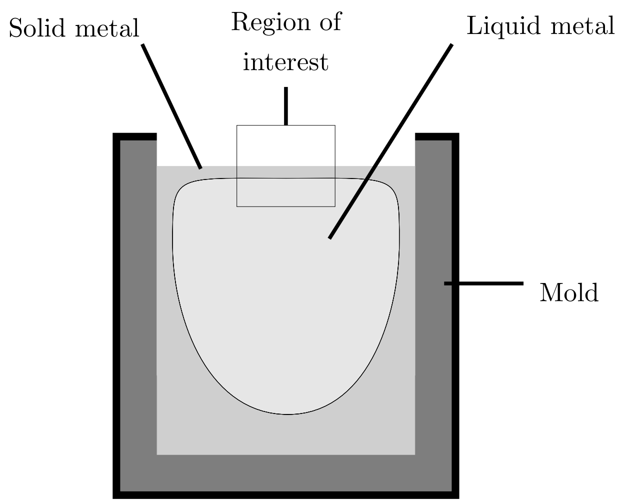
   (e) Assuming that solidification front velocity U is constant, that diffusion is slow enough that the diffusion boundary layer is much smaller than the part thickness, and that the effect of fluid flow is negligible, give the steady-state general equation for concentration as a function of distance into the liquid from the moving interface x' (that is, in the frame of reference of the moving interface). (4)

   (f) Fit this general solution to the boundary conditions in the liquid at $x' = 0$ and $x' = \infty$ to give the particular concentration profile here. (4)

   (g) Approximately how thick is the solute-enriched layer in the liquid (an expression, not a number)? (4)

2. Freezing by radiation and convection (26 pts)

   Castings with large open tops often "freeze off" by radiation and convection, forming a solid shell on top and trapping the liquid beneath it. (Because the liquid shrinks during solidification, this results in a large shrinkage cavity.) Here you will analyze the rate of solidification downward from the top surface in an low-carbon steel ingot casting due to these factors.



   Assume that the temperature is uniform, and for part 2b, the environment around the casting is gray along with the solid shell.

   Iron data:

   - Electrical conductivity near melting point: $\sigma = 5 \times 10^5 (\Omega \cdot \text{m})^{-1}$.
   - Wiedmann-Franz constant: $L = 2.45 \times 10^{-8} \frac{\text{W}\Omega}{\text{K}^2}$.
   - Density: $\rho = 7500 \frac{\text{kg}}{\text{m}^3}$.
   - Heat capacity: $c_p = 500 \frac{\text{J}}{kg \cdot K}$
   - Melting point: $T_m = 1800\text{K}$.
   - Heat of fusion: $\Delta H_f = 2.67 \times 10^5 \frac{J}{kg}$
   - Radiative emissivity: $\epsilon = 0.6$
   - Radiation constant: $\sigma = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4}$.
   - Heat transfer coefficient to air: $h = 100 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}$.

   (a) Estimate the thermal conductivity of iron near its melting point. (3)

   (b) Write an expression for the total radiative and convective heat flux from the top surface of the solidifying metal shell to the surrounding environment. (5)

   (c) Assuming the environment is much colder than the shell and absorbs all radiation (*i.e.* is "black"), calculate a total "heat transfer coefficient" which is the ratio between heat flux and absolute temperature. (5)

   (d) Use your heat transfer coefficient from part 2c to estimate the thickness of solid metal Y at which temperature can no longer be considered uniform (where the Biot number reaches 0.1). (4)

   (e) Estimate the rate of growth of the solid while solidification rate is limited by radiation/convection from the top surface (that is, while solid temperature can be considered uniform). How long does it take to reach the thickness calculated in part 2d? (5)

(f) Set up the equation for solidification rate limited by both radiation/convection from the top and also (quasi-steady-state) conduction through the solid (but don't try to solve it). (4)

3. Non-Newtonian polymer flow in a channel (28 pts)

A polymer melt is forced through a flat channel of thickness δ between two fixed horizontal plates by a pressure difference ΔP (inlet pressure minus outlet pressure). This channel length L is much longer than its width W, and it is a lot longer and wider than it is thick (δ), so you may assume fully-developed flow and neglect edge effects. Take the x direction to be the direction of flow, parallel to the plates, and the y direction to be straight up, perpendicular to the plates.

This polymer is a pseudoplastic non-Newtonian fluid, whose behavior can be modeled using a "power law":

$$\tau_{yx} = -\mu_0 \left|\frac{\partial u_x}{\partial y}\right|^{n-1} \frac{\partial u_x}{\partial y},$$

which is to say, shear stress is proportional to shear strain rate to the n power, with the extra $\partial u_x/\partial y$ there to get the sign right. The differential equation describing x-momentum equation for this laminar 1-D flow (where $u_y = u_z = 0$ and $d\vec{u}/dx = d\vec{u}/dz = 0$) can be written with the above power law substituted for τ_{yx}:

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial P}{\partial x} + \mu_0 \frac{\partial}{\partial y}\left(\left|\frac{\partial u_x}{\partial y}\right|^{n-1} \frac{\partial u_x}{\partial y}\right) + F_x.$$

The general solution to this equation for steady-state flow driven only by pressure gradient is:

$$u_x = -\frac{\mu_0 L}{\Delta P}\frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 L} y + C_1\right)^{\frac{n+1}{n}} + C_2.$$

Note that this is only valid for negative y; the positive y solution is symmetric.

(a) If the polymer is pseudoplastic (lower apparent viscosity at higher shear strain rate), is n greater or less than one? Explain your answer. (5)

(b) Determine the specific solution which fits the above general solution to the no-slip boundary conditions at the two stationary plates, and/or the symmetry plane boundary condition halfway between them (you need just two of these three conditions). Note: it might help to set $y = 0$ halfway between the plates. (12)

(c) Sketch this velocity profile for an appropriate pseudoplastic n value of your choosing. (6)

(d) For channel flow of this kind, a Newtonian fluid's average velocity will be 2/3 of its maximum velocity. For this pseudoplastic fluid, will the average velocity be more or less than 2/3 of its maximum? (5)

4. Plate glass casting (28 pts)

High-quality plate glass is cast with extremely flat surfaces by floating it on molten tin and letting it cool as the glass and tin flow down a gently inclined plane.

Data:

- Liquid glass entering the process: $\rho = 3.2\frac{\text{g}}{\text{cm}^3}$, $\mu = 1.0\frac{\text{N}\cdot\text{s}}{\text{m}^2}$, layer thickness=1mm.
- Liquid tin: $\rho = 7.0\frac{\text{g}}{\text{cm}^3}$, $\mu = 3 \times 10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2}$, layer thickness=1mm.
- Inclined plane angle: 2°.

(a) Write the general solution to the equation for momentum conservation down the inclined plane in the flow direction. (4)

(b) What is the interface boundary condition at the glass-tin interface? (4)

(c) Derive an expression for the velocity distribution in both the glass and the tin. (Hint: measure distance in the tin from the bottom, and distance in the glass from the top surface.) (9)

(d) What are the maximum and average velocities in the glass and tin? (7)

(e) Comment on the Reynolds numbers in the glass and tin. Do you think flow will remain laminar in both layers? (4)

5. Settling of magnesium hydroxide particles in water (17 pts)

Small and relatively uniform magnesium oxide particles can be made by precipitating magnesium hydroxide from a supersaturated aqueous solution and then drying it. Here we will study how quickly the $Mg(OH)_2$ particles sink through stagnant solution so they can be collected from the bottom of the container.

(a) Derive an expression for the terminal rising/sinking velocity of a sphere in a fluid, starting from the friction factor definition for flow past a sphere, the weight and buoyancy force, and the relation between f and Re in Stokes flow. (6)

(b) Consider a uniform suspension of approximate spheres of magnesium hydroxide ($\rho = 2360\frac{\text{kg}}{\text{m}^3}$) in water. If the water is 10 cm deep, above what diameter do all of the spheres reach the bottom within one minute? (That is, above what diameter is the terminal velocity at least 10 cm/minute?) (4)

(c) Now consider the magnesia particles with half that diameter. What fraction of those reach the bottom within one minute? (4)

(d) Check the Reynolds numbers at your critical velocity in part 5b to ensure Stokes flow is a valid assumption. (3)