4.2 March 14, 2005: Wrap up 1-D flows, drag force on a sphere

Mechanics:

- Evals Wednesday
- PS4 due 3/30
- Talk this Friday...

Nice segue into pressure-driven flows. Suppose fluid in a cylinder, a pipe for example of length L and radius R, P_1 on one end, P_2 on other. Net force: $(P_1-P_2)A_{xs}$, force per unit volume is $(P_1-P_2)V/A_{xs} = (P_1-P_2)/L$. Can shrink to shorter length, at a given point, force per unit volume is $\Delta P/\Delta z \to \partial P/\partial z$. This is the pressure generation term.

So, flow in tube: uniform generation throughout $(P_1 - P_2)/L$ (prove next week), diffusion out to $r = R$ where velocity is zero. Could do momentum balance, but is same as diffusion or heat conduction, laminar Newtonian result:

$$\rho\frac{\partial u_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u_z}{\partial r}\right) + \rho g_z - \frac{\partial P}{\partial z}. \tag{4.22}$$

Here looking at steady-state, horizontal pipe, uniform generation means:

$$u_z = -\frac{F_z r^2}{4\mu} + A\ln r + B = -\frac{P_1 - P_2}{4\mu L}r^2 + A\ln r + B. \tag{4.23}$$

Like reaction-diffusion in problem set 2 (PVC rod): non-infinite velocity at $r = 0$ means $A = 0$ (also symmetric), zero velocity at $r = R$ means:

$$u_z = \frac{P_1 - P_2}{4\mu L}(R^2 - r^2). \tag{4.24}$$

What's the flow rate?

$$Q = \int_0^R u_z 2\pi r dr = \int_0^R \frac{P_1 - P_2}{4\mu L}(R^2 - r^2)2\pi r dr \tag{4.25}$$

$$Q = \frac{\pi(P_1 - P_2)}{2\mu L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{\pi(P_1 - P_2)R^4}{8\mu L}. \tag{4.26}$$

Hägen-Poisseuille equation, note 4th-power relation is extremely strong! 3/4" vs. 1/2" pipe...

Summary of the three phenomena thus far:

	Diffusion	Heat conduction	Fluid flow
What's conserved?	Moles of each species	Joules of energy	kg m/s momentum
Local density	C	$\rho c_p T$	$\rho\vec{u}$
Units of flux	$\frac{\text{mol}}{\text{m}^2\cdot\text{s}}$	$\frac{\text{W}}{\text{m}^2}$	$\frac{\text{kg}\frac{\text{m}}{\text{s}}}{\text{m}^2\cdot\text{s}} = \frac{\text{N}}{\text{m}^2}$
Conservation equation*	$\frac{\partial C}{\partial t} = -\nabla\cdot\vec{J} + G$	$\rho c_p\frac{\partial T}{\partial t} = -\nabla\cdot\vec{q} + \dot{q}$	$\frac{\partial(\rho\vec{u})}{\partial t} = -\nabla P - \nabla\cdot\tau + \vec{F}$
Constitutive equation	$\vec{J} = -D\nabla C$	$\vec{q} = -k\nabla T$	$\tau = -\mu\left[\nabla\vec{u} + (\nabla\vec{u})^T\right]$
Diffusivity	D	$\alpha = k/\rho c_p$	$\nu = \mu/\rho$
Result**	$\frac{\partial C}{\partial t} = D\nabla^2 C + G$	$\frac{\partial T}{\partial t} = \alpha\nabla^2 T + \frac{\dot{q}}{\rho c_p}$	$\frac{\partial\vec{u}}{\partial t} = -\frac{\nabla P}{\rho} + \nu\nabla^2\vec{u} + \frac{\vec{F}}{\rho}$
Convective flux	$C\vec{u}$	$\rho c_p T\vec{u}$	$\rho\vec{u}\vec{u}$
New result**	$\frac{DC}{Dt} = D\nabla^2 C + G$	$\frac{DT}{Dt} = \alpha\nabla^2 T + \frac{\dot{q}}{\rho c_p}$	$\frac{D\vec{u}}{Dt} = -\frac{\nabla P}{\rho} + \mu\nabla^2\vec{u} + \frac{\vec{F}}{\rho}$

*Only considering diffusive fluxes. ^{T} in fluid constit. means matrix transpose. **For uniform properties.

New stuff: vector field instead of scalar; very different units; pressure as well as flux/shear stress and force.

Having taken 3.032, shear stress τ relates to stress σ as follows:

$$\sigma = -\tau - PI,\ P = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{4.27}$$

so $\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$, $\tau_{xy} = \tau_{yx} = -\sigma_{xy} = -\sigma_{yx}$. Note that $\tau_{yx} = \tau_{xy}$ almost always, otherwise infinite rotation...

Flow past a sphere Motivating process: Electron beam melting and refining of titanium alloys. Water-cooled copper hearth, titanium melted by electron beams, forms solid "skull" against the copper. Clean heat source, liquid titanium contained in solid titanium, results in very clean metal. Mystery of the universe: how does liquid Ti sit in contact with solid Cu? Main purpose: removal of hard TiN inclusions often several milimeters across which nucleate cracks and bring down airplanes! (1983 Sioux City, Iowa.)

Set up problem: sphere going one way u_{sphere}, fluid other way u_∞, local disturbance but relative velocity $U = u_\infty - u_{sphere}$, relative veloc of fluid in sphere frame. Drag force is in this direction.

For a sphere, drag force is slightly different: it has not only shear, but pressure component as well. Traction $\vec{t} = \sigma \cdot \hat{n}$. Stokes flow: ingore the convective terms, result:

$$F_d = 3\pi\mu dV. \tag{4.28}$$

At high velocity, friction factor concept:

$$F_d = fKA = f \cdot \frac{1}{2}\rho U^2 \cdot \frac{1}{4}\pi d^2. \tag{4.29}$$

Low Re (<0.1) means Stokes flow, can ignore all convective terms; analytical result in 3.21 notes, drag force:

$$F_d = 3\pi\mu U d = f \cdot \frac{1}{2}\rho U^2 \cdot \frac{1}{4}\pi d^2 \Rightarrow f = \frac{24\mu}{\rho U d} = \frac{24}{\mathrm{Re}}. \tag{4.30}$$

If faster, though not turbulent, f becomes a constant: about 0.44, that's the drag coefficient for a sphere. Cars as low as 0.17, flat disk just about 1, making dynamic pressure a good estimate of pressure difference.

Note: for bubbles, $F_d = 2\pi\mu U d$ all the way out to Re=10^5!

Reynolds number Low velocity: shear stress; high velocity: braking kinetic energy. Ratio of forces:

$$\mathrm{Re} = \frac{\text{convective momentum transfer}}{\text{shear momentum transfer}} = \frac{\text{inertial forces}}{\text{viscous forces}} \simeq \frac{\rho u_y \frac{\partial u_x}{\partial y}}{\mu \frac{\partial^2 u_x}{\partial y^2}} \simeq \frac{\rho U U/L}{\mu U/L^2} = \frac{\rho U L}{\mu}. \tag{4.31}$$