2.3 February 9, 2005: Loose ends, finite differences

Handout: finite differences made using pdftk of Continuum handout.

Cylindrical multilayer wall Slides, mention heat transfer coefficient on outside as resistance $1/h$. Final result:

$$q_x = \frac{T_1 - T_{fl}}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} + \frac{1}{h}}$$

Cylindrical: slightly different

$$Q = 2\pi r L q_r = \frac{2\pi L(T_1 - T_5)}{\frac{1}{k_1}\ln\frac{R_2}{R_1} + \frac{1}{k_2}\ln\frac{R_3}{R_2} + \frac{1}{k_3}\ln\frac{R_4}{R_3} + \frac{1}{R_4 h}}$$

Biot Number One layer and a heat transfer coefficient

$$q_x = \frac{T_1 - T_{fl}}{\frac{L}{k} + \frac{1}{h}}$$

$$\text{Bi} = \frac{\text{solid conduction resistance}}{\text{fluid BL resistance}} = \frac{hL}{k}$$

Also:

$$\text{Bi} = \frac{T_2 - T_1}{T_{fl} - T_2}$$

Unsteady Solutions Responsible for two:

- 1-D semi-infinite uniform initial, constant T boundary:

$$\frac{T - T_s}{T_\infty - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right). \tag{2.15}$$

  Example: two blocks of same material at different temperatures, come together (like a diffusion couple)

- 1-D infinite, uniform initial T, heat deposited at $x = 0$: Gaussian

$$T - T_i = \frac{(T_0 - T_i)\delta}{\sqrt{\pi\alpha t}}\exp\left(-\frac{x}{4\alpha t}\right). \tag{2.16}$$

  Example: resistance welding, brazing, some adhesives... Note $T_0\delta$ can be replaced with $H/\rho c_p$.

Even more on the handout, not responsible for any further than handout (and not asterisks either).

Finite differences Very often no analytical solution to a system. (Or if there is one, it's impossibly complex.) So, use a computer, make some approximations.

- Discretize space: calculate temperature at a finite number of points on a grid (here 1-D). Choose x_i, calculate T_i. For simplicity, we'll choose evenly-spaced points, so $x_{i+1} - x_i = \Delta x$.
- Discretize time: calcluate temperature at a finite number of "timesteps" at times t_n, so with both, we have $T_{i,n}$. For simplicity, Δt uniform.

- Make some approximations about derivatives:

$$\left.\frac{\partial T}{\partial t}\right|_{x_i,t_{n+1/2}} \simeq \frac{T_{i,n+1}-T_{i,n}}{\Delta t}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x_{i+1/2},t_n} \simeq \frac{T_{i+1}-T_i}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x_i,t_n} \simeq \frac{\left.\frac{\partial T}{\partial x}\right|_{x_{i+1/2},t_n} - \left.\frac{\partial T}{\partial x}\right|_{x_{i-1/2},t_n}}{\Delta x} \simeq \frac{T_{i-1,n}-2T_{i,n}+T_{i+1,n}}{(\Delta x)^2}$$

So, let's look at the energy equation, and substitute approximations:

$$\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

$$\frac{T_{i,n+1}-T_{i,n}}{\Delta t} = \alpha\frac{T_{i-1,n}-2T_{i,n}+T_{i+1,n}}{(\Delta x)^2} + \frac{\dot{q}}{\rho c_p}$$

$$T_{i,n+1} = T_{i,n} + \Delta t\left[\frac{T_{i-1,n}-2T_{i,n}+T_{i+1,n}}{(\Delta x)^2} + \frac{\dot{q}}{\rho c_p}\right] = T_{i,n} + \mathrm{Fo}_M(T_{i-1,n}-2T_{i,n}+T_{i+1,n} + \frac{\Delta t}{\rho c_p}\dot{q}$$

This is the "forward Euler" algorithm, a.k.a. "explicit" time stepping. Nice, efficient, easy to put in a spreadsheet. Problems: inaccurate because time and space derivatives not co-located, also unstable. Inaccuracy later.

Demonstrate instability for $\mathrm{Fo}_M > \frac{1}{2}$:

$$T_{i,n+1} = T_{i,n}(1-2\mathrm{Fo}_M) + 2\mathrm{Fo}_M\frac{T_{i-1,n}+T_{i+1,n}}{2} + \frac{\Delta t}{\rho c_p}\dot{q}$$

So, it's like a weighted average between $T_{i,n}$ and the average of the two (show graphically). When $\mathrm{Fo}_M > \frac{1}{2}$, the $T_{i,n}$ part is negative, so we shoot past it! So, the criterion is that it must be $\leq \frac{1}{2}$, larger timestep means less work, so use $\frac{1}{2}$.

Exercise: cut length step in half, for same total time, how many more timesteps? How much more computational work? Spreadsheet area...