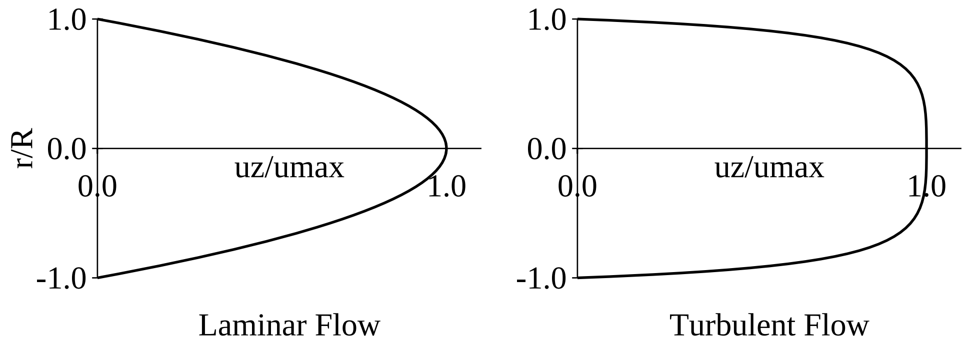
1. Turbulence and mixing in a tube

   (a) The laminar velocity profile is parabolic, and the turbulent profile looks somewhat like pseudo-plastic flow, since there's less mixing, and lower turbulent viscosity, and steeper velocity gradient near the tube walls away from the center. Your sketches should have looked something like:



   (b) Start by calculating the average velocity, to get the Reynolds number and friction factor:

$$\bar{u} = \frac{Q}{A_{xs}} = \frac{0.001\frac{\text{m}^3}{\text{s}}}{\pi \cdot \left(\frac{0.01\text{m}}{2}\right)^2} = 12.7\frac{\text{m}}{\text{s}}$$

$$\text{Re} = \frac{\bar{u}d}{\nu} = \frac{12.7\frac{\text{m}}{\text{s}} \cdot 0.01\text{m}}{10^{-6}\frac{\text{m}^2}{\text{s}}} = 1.27 \times 10^5$$

At this Reynolds number, and the dimensionless roughness $\epsilon/d = 10^{-5}\text{m}/10^{-2}\text{m} = 10^{-3}$, the graph provided gives a friction factor of approximately 5.5×10^{-3}.

Next we set the drag force magnitude equal to the pressure force, since they're equal and opposite, and solve for ΔP:

$$F_d = fKA_{tube} = \Delta P \cdot A_{xs}$$

$$f \cdot \frac{1}{2}\rho\bar{u}^2 \cdot 2\pi RL = \Delta P \cdot \pi R^2$$

$$\Delta P = \frac{L}{R} f\rho\bar{u}^2 = \frac{30\text{m}}{0.005\text{m}} \cdot 5.5 \times 10^{-3} \cdot 1000\frac{\text{kg}}{\text{m}^3} \cdot \left(12.7\frac{\text{m}}{\text{s}}\right)^2 = 5.35 \times 10^6 \frac{\text{N}}{\text{m}^2} (5.35\text{MPa})$$

Q.E.D.

   (c) The Hägen-Poiseuille equation states:

$$Q = \frac{\pi R^4 \Delta P}{8\mu L}$$

This is used for laminar flow, so in this turbulent case, it gives a good estimate of the average turbulent viscosity in this tube. Just substitute μ_t for μ and solve for μ_t:

$$\mu_t = \frac{\pi R^4 \Delta P}{8QL} = \frac{\pi \cdot (0.005\text{m})^4 \cdot 5.35 \times 10^6 \frac{\text{N}}{\text{m}^2}}{8 \cdot 0.001\frac{\text{m}^3}{\text{s}} \cdot 30\text{m}} = 0.0438\frac{\text{kg}}{\text{m} \cdot \text{s}}$$

So the turbulent viscosity is more than forty times the molecular viscosity of $10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}$.

   (d) Use the turbulent Prandtl number of one to estimate the turbulent diffusivity:

$$\text{Pr}_t = \frac{\nu_t}{D_t} = \frac{\mu_t}{\rho D_t} \simeq 1$$

$$D_t \simeq \frac{\mu_t}{\rho} = \frac{0.0438\frac{\text{kg}}{\text{m}\cdot\text{s}}}{1000\frac{\text{kg}}{\text{m}^3}} = 4.38 \times 10^{-5}\frac{\text{m}^2}{\text{s}}$$

This mixes across the tube in approximately the diffusion timescale of the diameter (or radius is okay):

$$t_{ss} = \frac{d^2}{D_t} = \frac{(0.01\text{m})^2}{4.38 \times 10^{-5}\frac{\text{m}^2}{\text{s}}} = 2.28\text{seconds}$$

This is the time required to mix (but not homogenize) the substance across the width of the tube, and is nearly the time required to travel the length of the tube!

(e) The power is the flow rate times pressure drop: power $= Q\Delta P$ and the energy dissipation rate per unit volume is simply the power dissipated divided by the volume: $\epsilon = \text{power}/V$. If we assume that all of the power goes into turbulent dissipation, this gives us:

$$\epsilon = \frac{Q\Delta P}{\pi R^2 L} = \frac{0.001\frac{\text{m}^3}{\text{s}} \cdot 5.35 \times 10^6\frac{\text{N}}{\text{m}^2}}{\pi \cdot (0.005\text{m})^2 \cdot 30\text{m}} = 2.27 \times 10^6 \frac{\text{N}}{\text{m}^2 \cdot \text{s}}$$

Since one newton-meter is a joule, and a joule per second is a watt, this is $2.27 \times 10^6\frac{\text{W}}{\text{m}^3}$.

(f) The expression for turbulent microscale was given on the equation sheet:

$$\ell \simeq \sqrt[4]{\frac{\mu^3}{\rho^2\epsilon}}$$

The dissipation rate ϵ (not the surface roughness ϵ) comes from part 1e, and the viscosity used is the *molecular* viscosity (not the turbulent viscosity) because molecular viscosity is turning mechanical power into heat energy in these small eddies, turbulent viscosity mixes the fluid at much larger lengthscales. So we have:

$$\ell \simeq \sqrt[4]{\frac{\left(10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}\right)^3}{\left(1000\frac{\text{kg}}{\text{m}^3}\right)^2 \cdot 2.27 \times 10^6\frac{\text{W}}{\text{m}^3}}} = \sqrt[4]{4.40 \times 10^{-22}\text{m}^4} = 4.6 \times 10^{-6}\text{m}$$

About five microns.

(g) Turbulence mixes the substance to the lengthscale in part 1f, then molecular diffusion has to homogenize it the rest of the way. The timescale for this second part is simply the steady-state diffusion time for the small eddies:

$$t_{homog} \simeq t_{ss} \simeq \frac{\ell^2}{D} = \frac{(4.6 \times 10^{-6}\text{m})^2}{10^{-6}\frac{\text{cm}^2}{\text{s}} \cdot \left(\frac{1\text{m}}{100\text{cm}}\right)^2} = 0.210\text{seconds}$$

So in this case, the substance is mixed across the tube in about two seconds, and diffuses on the eddy lengthscale it about two tenths of a second. It should be completely homogenized within about 2.5 seconds, which for fluid travelling 30 m at 12.7 m/s is about the amount of time the fluid spends in the tube.

(h) Since the mixing time across the tube dominates the time required for homogenization, we would like to understand conditions which would cut that mixing time in half. According to part 1d, that mixing time is inversely proportional to turbulent diffusivity D_t, which in turn is proportional to turbulent viscosity μ_t, so we want to double the turbulent viscosity.

The turbulent viscosity, in turn, is proportional to the ratio of pressure gradient to flow rate $\Delta P/QL$, so we want to find conditions which would double that ratio. Flow rate is proportional to average velocity, and at roughly constant friction factor, pressure gradient is proportional to average velocity squared. Thus doubling the average velocity would approximately double $\Delta P/QL$, μ_t and D_t. This would require that the pressure gradient increase by about a factor of four, or double the pressure drop ($\sim$ 10 MPa) over half the length.