# 3.044 Problem Set 2

## Advanced Heat Conduction

## Solutions

1. Spreadsheet finite difference model of 1-D unsteady conduction

   (a) The Biot number is given by:

   $$\frac{hL}{k} = \frac{9000\frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}} \cdot 5 \times 10^{-3}\mathrm{m}}{0.21\frac{\mathrm{W}}{\mathrm{m}\cdot\mathrm{K}}} = 214$$

   Because this is well above 100, we can use the constant-temperature boundary condition approximation.

   Note: even if you used half the thickness for $L$, your Biot number would still be above 100.

   (b) The 1-D explicit finite difference method begins with:

   $$\frac{T_{i,n+1} - T_{i,n}}{\Delta t} = \alpha \frac{T_{i-1,n} - 2T_{i,n} + T_{i+1,n}}{(\Delta x)^2}$$

   where the subscripts give the $x$ node number and timestep number respectively. Solving for $T_{i,n+1}$ gives:

   $$T_{i,n+1} = T_{i,n} + \frac{\alpha \Delta t}{(\Delta x)^2}(T_{i-1,n} - 2T_{i,n} + T_{i+1,n})$$

   The quantity $\alpha\Delta t/(\Delta x)^2$ is the mesh Fourier number $\mathrm{Fo}_M$, so we can rewrite this as:

   $$T_{i,n+1} = (1 - 2\mathrm{Fo}_M)T_{i,n} + \mathrm{Fo}_M(T_{i-1,n} + T_{i+1,n})$$

   (c) Since there are seven intervals, the $x$-spacing between nodes is one-seventh of the thickness, or $\frac{5}{7}$mm. The maximum allowed mesh Fourier number is 1/2, so this gives us the maximum allowed timestep as follows:

   $$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

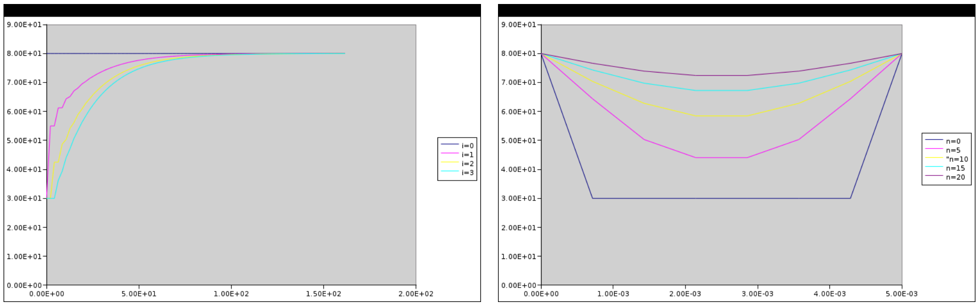
   $$\Delta t \leq \frac{(\Delta x)^2}{2\alpha}$$

   Here $\alpha = k/(\rho c_p) = 1.2 \times 10^{-7}\frac{\mathrm{m}^2}{\mathrm{s}}$, so this maximum timestep is:

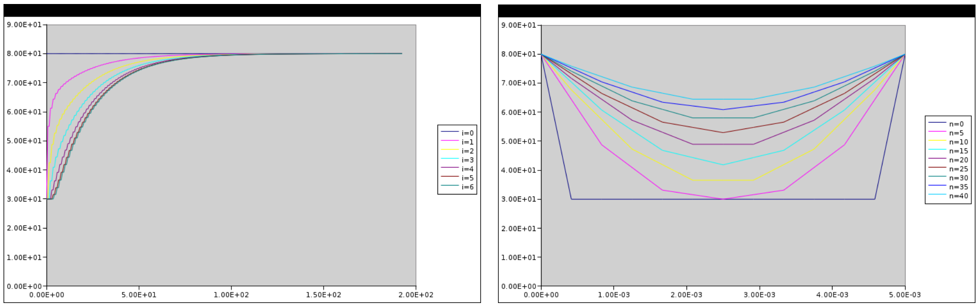
   $$\Delta t \leq \frac{\left(\frac{5}{7} \times 10^{-3}\mathrm{m}\right)^2}{2 \cdot 1.2 \times 10^{-7}\frac{\mathrm{m}^2}{\mathrm{s}}} = 2.13\mathrm{s}$$

   With 12 intervals, $\Delta x$ drops below 1/2mm, so the maximum timestep falls to 0.72 seconds.

(d) You could either graph this in terms of time with three plots at $x_0$, $x_1$ and $x_2$ (since the other three are symmetric), or with respect to $x$ at several timesteps. Both are shown below:



(e) Same as the last part, though there are more data:



Interestingly, with an even number of nodes and at the critical timesteps size ($\text{Fo}_M$=0.5), the temperature profiles are less smooth, and with time, each node only changes every other timestep.

2. Radiative cooling of an aluminum cube

(a) The radiative flux from the surface in a cold black enclosure is simply

$$q_{rad} = e = \epsilon\sigma T^4.$$

(b) With an "environment temperature" of zero, we set:

$$q_{rad} = h_{rad}(T-0) = \epsilon\sigma T^4 \Rightarrow h_{rad} = \epsilon\sigma T^3.$$

(c) Since the size and thermal conductivity are constant, the "maximum Biot number" over this temperature range corresponds to the Biot number for maximum radiative "heat transfer coefficient" $h_{rad}$. Since $h_{rad}$ is an increasing function of temperature, this will be at the highest temperature, 1000K.

$$\text{Bi}_{rad} = \frac{0.85 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^4 \cdot \text{K}^4} \cdot (1000\text{K})^3 \cdot 0.1\text{m}}{238 \frac{\text{W}}{\text{m} \cdot \text{K}}} = 0.020$$

Since the maximum Biot number is well below 0.1, the uniform temperature assumption is valid.

(d) This setup begins as a typical Newtonian cooling problem, but you've never quite seen it in this form before, so you have to solve a new differential equation:

$$\text{accumulation} = -\text{out}$$

$$V\rho c_p \frac{dT}{dt} = Aq = A\epsilon\sigma T^4$$

$$\frac{V\rho c_p}{A\epsilon\sigma}\frac{dT}{T^4} = dt$$

$$\frac{V\rho c_p}{A\epsilon\sigma}\int_{T_{init}}^{T_{final}}\frac{dT}{T^4} = \int_{t_{init}=0}^{t_{final}} dt$$

$$\frac{V\rho c_p}{A\epsilon\sigma}\left[-\frac{1}{3T^3}\right]_{T_{init}}^{T_{final}} = t_{final} - 0$$

$$t_{final} = \frac{V\rho c_p}{3A\epsilon\sigma}\left(\frac{1}{T_{final}^3} - \frac{1}{T_{init}^3}\right)$$

$$t_{final} = \frac{(0.1\text{m})^3 \cdot 2700\frac{\text{kg}}{\text{m}^3} \cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}}{3 \cdot 5 \cdot (0.1\text{m})^2 \cdot 0.85 \cdot 5.67 \times 10^{-8}\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}}\left(\frac{1}{(400\text{K})^3} - \frac{1}{(1000\text{K})^3}\right) \simeq 5000\text{seconds}$$

Note that at 400K, if the surroundings are at room temperature (aronud 300 K) and the room and cube can be considered "grey" surfaces, they return $(3/4)^4 = 32\%$ of the radiative flux back to the cube, so the flux is quite a bit lower—and cooling is quite a bit slower—by radiation alone. On the other hand, convection becomes more important at the lower temperatures.

Note that this analysis captures the enormous reduction in heat loss rate as the cube cools, which is not accounted for in the constant-$h$ Newtonian cooling formulation. Pretty neat, huh?

3. Dimensional analysis: resistance welding

   The Gaussian solution to the time-dependent diffusion equation:

$$T = T_{init} + \frac{\beta}{\rho c_p\sqrt{\pi D t}}\exp\left(-\frac{x^2}{4Dt}\right)$$

   (a) Dimensions and units:

| Dimension | Units |
|---|---|
| $T - T_{init}$ | K |
| $\beta$ | $\frac{\text{J}}{\text{m}^2}$ |
| $\rho c_p$ | $\frac{\text{J}}{\text{m}^3\cdot\text{K}}$ |
| $\alpha$ | $\frac{\text{m}^2}{\text{s}}$ |
| $t$ | seconds |
| $x$ | meters |

   Notes: we can also combine $\beta$ and $\rho c_p$ into $\beta/\rho c_p$ to have five dimensions instead of six; its units are $\text{K}\cdot\text{m}$.

   (b) From above, there are six dimensions (five if we use $\beta/\rho c_p$), and four base units J, K, m, s (omit J to leave three if using $\beta/\rho c_p$); either way, there are just two dimensionless parameters.

   (c) We want to construct $\pi_T$ and $\pi_x$, eliminating $\beta$, $\rho c_p$ $\alpha$ and $t$.

| Dimension | J | K | m | s |
|---|---|---|---|---|
| $T - T_{init}$ | 0 | 1 | 0 | 0 |
| $\rho c_p$ | 1 | –1 | –3 | 0 |
| $\beta^{-1}$ | –1 | 0 | 2 | 0 |
| $\alpha^{1/2}$ | 0 | 0 | 1 | $-\frac{1}{2}$ |
| $t^{1/2}$ | 0 | 0 | 0 | $\frac{1}{2}$ |
| Total | 0 | 0 | 0 | 0 |

| Dimension | J | K | m | s |
|---|---|---|---|---|
| $x$ | 0 | 0 | 1 | 0 |
| $\alpha^{-1/2}$ | 0 | 0 | –1 | $\frac{1}{2}$ |
| $t^{-1/2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ |
| $\beta^0$ | 0 | 0 | 0 | 0 |
| $(\rho c_p)^0$ | 0 | 0 | 0 | 0 |
| Total | 0 | 0 | 0 | 0 |

So we have:

$$\pi_T = (T - T_{init})\sqrt{\alpha t}\frac{\rho c_p}{\beta}$$

$$\pi_x = \frac{x}{\sqrt{\alpha t}}$$

(d) Start with the solution itself:

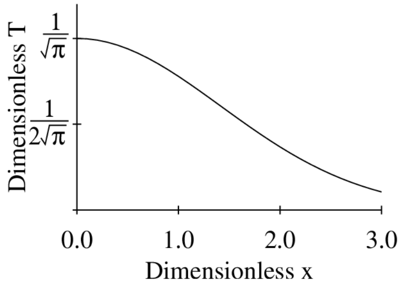
$$T = T_{init} + \frac{\beta}{\rho c_p \sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

One rearrangement should do it:

$$(T - T_{init})\sqrt{\alpha t}\frac{\rho c_p}{\beta} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4Dt}\right)$$

$$\pi_T = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\pi_x^2}{4}\right)$$

(e) When made dimensionless like this, the "Shrinking Gaussian" becomes a simple Gaussian with a maximum value of $1/\sqrt{\pi}$:



As for the width, you could define any measure of it, and give the corresponding value. One such measure is where the Gaussian reaches $1/e$ of its maximum; this is relatively easy because you just set the exponent to $-1$, so $-\pi_x^2/4 = -1$ and $\pi_x = 2$.

Another popular measure for Gaussian distributions is the "full width at half maximum" (FWHM). For this, you solve:

$$\exp\left(-\frac{\pi_x^2}{4}\right) = \frac{1}{2}$$

$$\pi_x = 2\sqrt{\ln 2} \simeq 1.39$$

Either of these measures was fine.

4. Polymer extrusion and thermal stress

(a) The Biot number is given by:

$$\mathrm{Bi} = \frac{hR}{k} = \frac{130\frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}} \cdot 0.01\mathrm{m}}{\frac{\mathrm{W}}{\mathrm{m}\cdot\mathrm{K}}} = 2.03$$

(b) For the Fourier number, we first convert $z$ to time:

$$u_z = \frac{z}{t}, \text{ so } t = \frac{z}{u_z}$$

Fourier number definition:

$$\mathrm{Fo} = \frac{\alpha t}{R^2} = \frac{kt}{\rho c_p R^2}$$

The length scale here is the radius; this was shown on the graphs handed out in class.

| Distance $z$ | time $t$ | Fourier number |
|---|---|---|
| 0.33 m | 3.3 s | $\sim 0.01$ |
| 1.0 m | 10 s | $\sim 0.03$ |
| 3.3 m | 33 s | $\sim 0.1$ |

(c) Since the Biot number is about 2, we use the corresponding curve in the graph with $r/R = 0$ (center) and $r/R = 1$ (surface), which give the dimensionless temperatures and real temperatures as follows:

| Distance | Fourier number | Center $\frac{T-T_f}{T_i-T_f}$ | Center T | Surface $\frac{T-T_f}{T_i-T_f}$ | Surface T |
|---|---|---|---|---|---|
| 0.33 m | 0.01 | 1.0 | 160°C | 0.8 | $\sim$ 135°C |
| 1.0 m | 0.03 | 1.0 | 160°C | 0.7 | $\sim$ 135°C |
| 3.3 m | 0.10 | 0.95 | $\sim$ 155°C | 0.5 | 100°C |

The third of these, at $z = 3.3$m, obviously has the largest temperature difference.

(d) This is a Biot number issue, since lower Biot numbers correspond to more uniformity in the solid. Indeed, for a Biot number below 0.1, we have a “Newtonian cooling” a.k.a. “lumped parameter” situation where temperature is approximately uniform.

To reduce the Biot number $\frac{hR}{k}$, we can:

- use a different material with higher thermal conductivity $k$, though that would require, well, using a different material, but we have orders for HDPE rods.
- reduce the radius $R$, though that would require, well, reducing the radius, but we have orders for 2 cm diameter rods.
- reduce the heat transfer coefficient by turning off or slowing down some or all of the cooling fans, though this would require longer cooling time for the extruded rods, and therefore a longer line in the factory or a slower production rate. Either way, the product will be more costly, but that’s better than shipping bad or out-of-spec product.

It’s probably obvious that I was looking for the last of these three answers, but the question wording was vague enough that any of them would do.

Note however that slowing down the line alone would *not* improve temperature uniformity, it would just, well, slow down the line.