Recitation Questions on Dimensional Analysis

Jorge Vieyra

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1 How to select the variables for r

Let's go through the example seen in class again:

$$q = \frac{\Delta T}{\frac{l}{k} + \frac{1}{h}} \quad (1)$$

1. First we postulate functional dependence and determine number of variables, n

$$q = f\left(\Delta T, h, l, K\right) \quad (2)$$

   Therefore, $n = 5$.

2. Determine base units, fundamental dimensions and number of **dimensionally-independent** variables, r.

   This time let's break all the units instead of keeping units of power (W).

$$\begin{aligned}
&[q] : \frac{W}{m^2} \rightarrow \frac{J}{sm^2} \rightarrow \frac{Nm}{sm^2} \rightarrow \frac{kg}{s^3} \rightarrow Mt^{-3} \\
&[\Delta T] : K \rightarrow T \\
&[h] : \frac{W}{m^2K} \rightarrow Mt^{-3}T^{-1} \\
&[l] : m \rightarrow L \\
&[k] : \frac{W}{mK} \rightarrow Mt^{-3}LT^{-1}
\end{aligned}$$

   Where M denotes units of Mass, t units of time, T units of Temperature and L units of Length.

   Now we have the base units, 4 base units. However NOT all the variables are dimensionally-independent!

   It is clear that T and l are independent. But if we examine carefully, we will see that the units of q are very similar to the units of k and h. Actually if we multiply the units of q by T^{-1} we will get the units of h and if we further multiply by units of l we will get units of k.

   Therefore we only have 3 dimensionally-independent variables, so $r = 3$.

Now, which variables should we keep?

Since we know that ΔT and l are dimensionally-independent the have definitely have to eliminate those them. You could also group h, k, $/DeltaT$ but let's focus on the other two.

We need to eliminate one more, so which one k or h? Well, pick the one you like and if you do both probably you would prefer the way the equations look using k. The variables are chosen in such a way that is convenient and useful for engineering purposes.

3. From the Buckingham Pi theorem we have that $n - r = 2$ so, we have 2 dimensionless groups, but we already new that :).

   For simplicity I am going to group Mt^{-3} as one compound unit since they always appear together.

$$\Pi_q = [q][\Delta T]^a[L]^b[k]^c$$

   The powers would be:

$$\begin{aligned} Mt^{-3} : 0 &= 1 + 0 + b + 0 \\ T : 0 &= 0 + a - b + 0 \\ L : 0 &= 0 + 0 + b + c \end{aligned}$$

   Therefore, $a = -1$, $b = -1$, $c = 1$. Not surprisingly, The same we result that we got in class!! :)

$$\Pi_h = \frac{hL}{k}$$

   But hey!! we have seen this Pi group before! It's the Biot number!!

   The dimensionless numbers are just Pi groups named after the first person who recognized it's importance and wanted to be famous.

2 Other Examples

2.1 Usefulness of the dimensionless Numbers

The easiest way to understand the meaning of these numbers is to see them as a ratio of quantities of interest.

For example:

Biot - This number is proportional to:

$$\frac{\text{(surface film conductance)}}{\text{(thermal internal conductance)}} = \frac{h}{k/L}$$

. Therefore a large Biot number means that heat is transfered faster on the surface than inside the material i.e. The center is hotter (or colder) than the edges. A small Biot number means that not much is happening on the surface and basically that the material is cooling (or heating) uniformly. It is used in heat transfer in general and unsteady state calculations in particular.

Fourier - ratio of current time to time to reach the steady state or

$$\frac{\alpha}{L^2/t}$$

. In other words how fast is the heat conduced inside a material. It gives Dimensionless time.

Reynolds - This number is proportional to:

$$\frac{\text{(convection)}}{\text{(viscous force)}} = \frac{L\upsilon\rho}{\eta}$$

(characteristic length) and is used in momentum, heat, and mass transfer processes. It is used to determine the flow regime in fluid dynamics, from laminar, transitional and turbulent regimes.

2.1.1 Amorphous metal

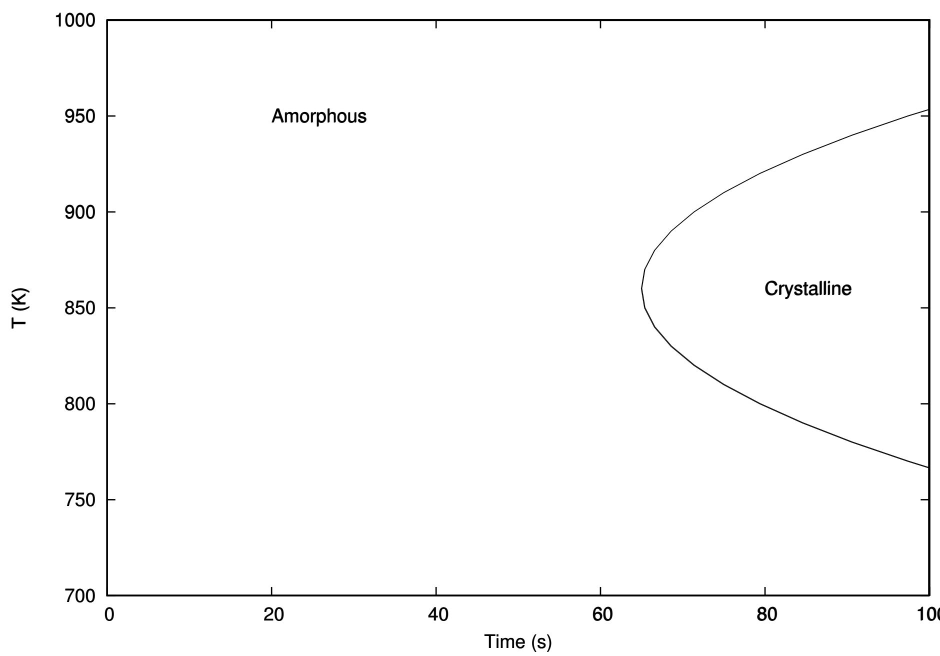
An amorphous metal is one which doesn't have a crystalline structure, this is done by quenching the metal faster than the time it takes to form a crystalline phase.

Originally this phenomenon was seen in the lab only in thin films.

A particular alloy called Vitreloy was studied, because you could make sheets up to 2cm thick.

So how fast should we cool it down to get an amorphous structure.

If we see the TTT curve below:



We can see that if cool it down faster than 60s when it reaches 860K the formed structure is amorphous.

So, is this material good enough for producing 2cm thick slabs? Well, we can answer that question using our favorite dimensionless numbers, Biot and Fourier number.

Fourier number tells us how fast heat is diffused in the material and Biot number just tells us if heat is conduced faster at the interface than it is conducted through the material.

$$\text{Fo} = \frac{\alpha}{L^2/t}$$

$$\text{Bi} = \frac{h}{k/L}$$

This particular alloy has a thermal conductivity of 10 $\text{Wm}^{-1}\text{K}^{-1}$ and a thermal diffusivity of $3.2 \times 10^{-6}\text{m}^2\text{s}^{-1}$. The sheet temperature is originally at 1000K and it is quenched down to room temperature. The heat transfer coefficient is $1500\text{Wm}^{-2}\text{K}^{-1}$.

So, if we want to get the time in which it is cooled below 860K Fourier number may be useful using the plots in Fig 9.8 from P&G.

But first we need the Biot number, which is:

$$Bi = \frac{(1500)(2 \times 10^{-2})}{10} = 3$$

Well, it's an intermediate number and that was kind of expected since we want to transfer all the heat through the interface to cool it down, but at the same time want to cool down all the sheet uniformly, therefore the materials were chosen for that, right?

If we want to use Fig 9.8, we need to calculate the dimensionless temperature.

$$\frac{T - T_f}{T_i - Tf} = \frac{860 - 300}{1000 - 300} = 0.8$$

So now that we have the curve at $Bi = 3$ and $\frac{T-T_f}{T_i-Tf} = 0.8$ we can determine from Fig. 9.8 that the Fourier number $Fo \approx .25$.

So, using the definition of Fo we get that the slab cools down in:

$$\text{t} = \frac{Fo \times L^2}{\alpha} = \frac{(0.25)(2 \times 10^{-2})^2}{3.2 \times 10^{-6}} = 31.5s$$

We needed it to be below 60s and we are cooling the plate in half that time, so we can form a 2cm thick sheet of amorphous metal with vitreloy!!!

2.2 The pendulum

The easiest example to explain Dimensional Analysis is the swinging pendulum. What are the variables that control the period T of the pendulum? This example is not in the context of the class, but I thought that it could probably be useful.

Well, we can deduce that it depends on length l and gravity g.

So the functional dependence would be.

$$T = f(l, g) \tag{3}$$

We have $n = 3$.

The relevant dimensions are $[T] = t$, length L and acceleration Lt^{-2}. Which are both independent so $r = 2$, so the problem has one dimensionless group.

$$\Pi_T = T l^{\alpha} g^{\beta} \tag{4}$$

To equate the exponents of the variables we find that:

$$1 - 2\beta = 0,$$
$$\alpha + \beta = 0$$

Which gives us $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$. Then:

$$\Pi_T = T\sqrt{\frac{g}{l}}$$

The number $\sqrt{\frac{g}{l}}$ would probably look familiar.

If we remember the differential equation:

$$\theta'' + \omega^2 sin(\theta) = 0$$
$$\omega^2 = \frac{g}{l}$$

Where ω is the frequency and we know that $\omega = \frac{2\pi}{T}$.