March 11, 2005: Newtonian Shear Flow II

Recap last time: momentum conservation, vector quantity. Shear stress as momentum flux; Newtonian viscosity.

Non-Newtonian fluids: liquid polymers, semi-solid slurries from metals to ceramic-binder systems, both have decreasing apparent viscosity with increasing shear.

Flow between parallel plates:

- Last time: steady-state, no generation, bottom velocity zero, top U:

$$u_x = \frac{U}{L}z,\ \tau_{zx} = -\mu\frac{U}{L} \tag{4.11}$$

  Shear stress:

$$\tau_{zx} = -\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = -\mu\frac{U}{z} \tag{4.12}$$

  Thus the drag force is this times the area of the plate.

- Unsteady, no generation, different velocities from $t = 0$.

$$u_x = U\text{erfc}\left(\frac{L-z}{2\sqrt{\nu t}}\right), \tau_{zx} = -\frac{1}{2\sqrt{\nu t}}\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(L-z)^2}{4\nu t}\right) \tag{4.13}$$

- New: steady-state, generation, with θ the inclination angle off-normal so $g_x = g\sin\theta$, z is the distance from the plane. The steady-state equation reduces to:

$$0 = \mu\frac{\partial^2 u_x}{\partial z^2} + F_x \tag{4.14}$$

$$u_x = -\frac{F_x z^2}{2\mu} + Az + B \tag{4.15}$$

  BCs: zero velocity at bottom plate at $z = 0$, free surface with zero shear stress at $z = L$, $F_x = \rho g_x = \rho g\sin\theta$, result: B=0, get

$$u_x = \frac{g\sin\theta}{2\nu}(2Lz - z^2) \tag{4.16}$$

  Shear stress:

$$\tau_{zx} = -\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \rho g\sin\theta(L-z) \tag{4.17}$$

  This is the weight of the fluid per unit area on top of this layer!

Shear stress as the mechanism of momentum flux, each layer pushes on the layer next to it. Think of it as momentum diffusion, not stress, and you'll get the sign right.

Flow rate: Q, volume per unit time through a surface. If width of the falling film is W, then flow rate is:

$$Q = \int_S \vec{u}\cdot\hat{n}dA = \int_{z=0}^{L} u_x W dz = \frac{Wg\sin\theta}{\nu}\left[\frac{Lz^2}{2} - \frac{z^3}{6}\right] = \frac{Wg\sin\theta}{\nu}\frac{L^3}{3} \tag{4.18}$$

Average velocity is Q/A, in this case Q/LW:

$$u_{av} = \frac{Q}{LW} = \frac{g\sin\theta L^2}{3\nu}; \tag{4.19}$$

$$u_{max} = u_x|_{z=L} = \frac{g\sin\theta L^2}{2\nu}. \tag{4.20}$$

So average velocity is 2/3 of maximum for falling film, channel flow, etc.

Also: Mechanics uses displacement for $\vec{u}$, acceleration is its *second* derivative with time. Simple shear:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \sigma + \vec{F} \Rightarrow \rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x^2} + \frac{\partial \sigma_{yx}}{\partial y^2} + \frac{\partial \sigma_{zx}}{\partial z^2} + F_x = G \frac{\partial^2 u_x}{\partial z^2} + F_x. \quad (4.21)$$

Analogue to momentum diffusivity: G/ρ, units $\mathrm{m}^2/\mathrm{s}^2$, $\sqrt{G/\rho}$: speed of sound! (Well, speed of transverse waves.) Remember with a little jig:

Fluids are diffusive,
With their velocity and viscosity.
But on replacement with displacement,
it will behave, like a wave!