3.044 Problem Set 5

Fluids and Processing

Solutions

1. Fluid flow in blood vessels

   (a) As a rough ballpark estimate, say about 6 ml of blood is pumped with every heartbeat. At a pulse of 80 beats/minute, the flow rate is 8 $\frac{\text{cm}^3}{\text{s}}$.

   (b) Average velocity is flow rate divided by cross-section area, which is $8\frac{\text{cm}^3}{\text{s}} \div [\pi(0.35\text{cm})^2]$, or about 21 $\frac{\text{cm}}{\text{s}}$.

   Reynolds number is $\frac{\bar{u}D}{\nu}$, $\bar{u} = 21\frac{cm}{s}$, $D = 0.7$cm, $\nu = 0.01\frac{cm^2}{s}$ for water, so it is about 1450. Flow is therefore laminar.

   (c) For laminar flow, we can use the Hägen-Poiseuille equation for flow rate in a tube, derived in class:

$$Q = \frac{P_1 - P_2}{L}\frac{\pi}{8\mu}R^4.$$

   We know the flow rate, viscosity and radius, so we can solve for the pressure gradient:

$$\frac{P_1 - P_2}{L} = \frac{8Q\mu}{\pi R^4},$$

   and then using the force balance

$$\pi R^2(P_1 - P_2) = 2\pi RL\tau_{rz} \Rightarrow \tau_{rz} = \frac{P_1 - P_2}{L}\frac{R}{2},$$

   we plug in the above equation for $(P_1 - P_2)/L$ to give:

$$\tau_{rz} = \frac{8Q\mu}{\pi R^4}\frac{R}{2} = \frac{4Q\mu}{\pi R^3}.$$

   Plugging in our values $Q = 8\frac{\text{cm}^3}{\text{s}} = 8 \times 10^{-6}\frac{\text{m}}{\text{s}}$, $\mu = 10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2}$, $R = 0.035$m gives

$$\tau_{\text{rz}} = \frac{4 \cdot 8 \times 10^{-6}\frac{\text{m}^3}{\text{s}} \cdot 10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2}}{\pi \cdot (0.0035\text{m})^3} = 0.24\text{Pa}.$$

   You could also start with the friction factor, which is $\frac{16}{\text{Re}} = 0.011$. Because we don't have the length, we can only approximate shear stress, which is:

$$\tau_0 = fK = f\left(\frac{1}{2}\rho\bar{u}^2\right) = \frac{0.011}{2} \cdot 1000\frac{\text{kg}}{\text{m}^3} \cdot \left(0.21\frac{\text{m}}{\text{s}}\right)^2 = 0.24\text{Pa}.$$

   If you used a larger ventricle volume or higher pulse rate it part 1a, the flow would be transitional or turbulent. In that case, you could use the friction factor analysis.

(d) If flow is laminar, we can extend the equation for the pressure drop $P_1 - P_2$ derived above:

$$\frac{P_1 - P_2}{L} = \frac{8Q\mu}{\pi R^4} = \frac{8\bar{u}\mu}{R^2}.$$

With the parameters of this part of the problem $\bar{u} = 10^{-3}\frac{\text{m}}{\text{s}}$, $R = 2.5 \times 10^{-5}\text{m}$, $\mu = 0.001\frac{\text{N}\cdot\text{s}}{\text{m}^2}$, this gives us a pressure gradient of

$$\frac{P_1 - P_2}{L} = \frac{8 \cdot 10^{-3}\frac{\text{m}}{\text{s}} \cdot 0.001\frac{\text{N}\cdot\text{s}}{\text{m}^2}}{(2.5 \times 10^{-5}\text{m})^2} = 12800\frac{\text{Pa}}{\text{m}}.$$

Multiplying by the length (1mm) gives 12.8 Pa for the pressure difference.

To use the friction factor, first calculate the Reynolds number which is $\frac{\bar{u}D}{\nu} = 0.05$, so flow is definitely laminar, and $f = 320$. This gives a shear stress of 0.16 Pa, and total drag force is this times the surface area of $1.57 \times 10^{-7}m^2$, so $F_d = 2.51 \times 10^{-18}N$, and $\Delta P = 12.8$ Pa.

(e) There are several important reasons why the above analysis will be inaccurate:

- The most important problem is that it assumes steady-state flow, which is obviously far from true.
- The tube flow analysis assumes rigid walls, whereas blood vessels are softenough to give slightly during each surge of pressure.
- The walls of blood vessels are not smooth, though this is much more of a factor for small blood vessels than large ones.
- For capillaries where blood cell size is a significant fraction of the diameter, the Newtonian flow assumption breaks down.
- For the aorta in particular, it is not a straight tube, but has a large bend, and dynamic pressure in that bend can be quite high. It is also short relative to the diameter, so fully-developed flow is questionable. And there are blood vessels branching off very early.

2. Turbulent microscale in an aqueous mixture

(a) We'll assume that all of the energy put into the mixer is dissipated in the smallest eddies of the turbulent flow. (A small amount will go into noise, and much of it will result in heat in the motor, but that's accounted for in the efficiency.)

$$\epsilon = 50\text{W} \times 75\% \div 1\text{liter} \times \frac{1000\text{liter}}{1\text{m}^3} = 37,500\frac{\text{W}}{\text{m}^3}$$

(b) The turbulent microscale can be derived by solving two equations for two unknowns: the equations are for approximate viscous dissipation and unit Reynolds number in the smallest eddies. The resulting expression for approximate lengthscale looks like:

$$\ell \simeq \sqrt[4]{\frac{\eta^3}{\rho^2\epsilon}}$$

For this problem, using epsilon from part 2a and the density and viscosity of water, we get

$$\ell \simeq \sqrt[4]{\frac{\left(10^{-3}\frac{kg}{m\cdot s}\right)^3}{\left(10^3\frac{kg}{m^3}\right)^2 \cdot 3.75 \times 10^4 \frac{kg}{m\cdot s^2}}} = 1.28 \times 10^{-5}\text{m}$$

About thirteen microns!

Assumptions: turbulence is isotropic and uniform. Good mixers are designed to approach uniform dissipation, but the fourth-root dependence means that ℓ is a lot more uniform than ϵ.

(c) The diffusion timescale across the smallest eddies is $\frac{\ell^2}{D}$, which is about 0.1 second. Because this is so small, homogenization is probably limited by factors like starting the mixer and transport of turbulence through the fluid.

With some more information we could estimate the thickness of the laminar sublayer—that stubborn layer near the edges of the bowl that you have to scrape with the mixer blades because it never quite gets blended in on its own.

(d) The timescale in either case is $\frac{L^2}{\nu}$, where $\nu = 10^{-6}\frac{\text{m}^2}{\text{s}}$ for water.

For the smallest eddies, we use $L = \ell$ and get $t \simeq 10^{-4}$ seconds.

For the main rotation pattern, the shortest dimension is at most about 0.1m (if the "bowl" is approximated as a cube), so $t \simeq 10^4$ seconds, or about three hours! The problem with this is that it assumes laminar flow, whereas there is significant eddy mixing for quite some time, and the turbulent viscosity is likely to be orders of magnitude larger than the molecular viscosity of water. So this is definitely an overestimate.

3. New tundish design

(a) Average velocity is flow rate divided by total cross section area of the four channels:

$$u_{av} = \frac{Q}{A} = \frac{0.002\frac{\text{m}^3}{\text{s}}}{4 \times 0.15\text{m} \times 1\text{m}} = 0.0033\frac{\text{m}}{\text{s}}.$$

(b) In the channels:

$$\text{Re} = \frac{\rho u_a v W_c}{\eta} = \frac{7000\frac{\text{kg}}{\text{m}^3} \cdot 0.00333\frac{\text{m}}{\text{s}} \cdot 0.15\text{m}}{5.2 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}} = 673.$$

Without the channels, the cross-section area is $W \times H$ which is one square meter, so the average velocity is 0.002 m/s. The Reynolds number in this situation is:

$$\text{Re} = \frac{7000\frac{\text{kg}}{\text{m}^3} \cdot 0.002\frac{\text{m}}{\text{s}} \cdot 0.1\text{m}}{5.2 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}} = 2692.$$

The Reynolds number is considerably higher in the open tundish without channels, and more importantly, crossing the 1000-2000 range makes it much more likely to be turbulent without the channels than with them.

(c) In well-behaved channel flow between parallel plates (laminar, fully-developed, steady-state, no edge effects, etc.), average velocity is two-thirds of maximum velocity, so maximum velocity is

$$u_{max} = \frac{3}{2}u_{av} = 0.005\frac{\text{m}}{\text{s}}.$$

(d) To ensure removal, the ceramic particle must rise from the bottom to the top of the tundish during the minimum time it spends channel. The minimum time is determined by the maximum velocity

$$t_{min} = \frac{L}{u_{y,max}},$$

so the minimum upward particle velocity needed to ensure removal is:

$$u_{z,min} = \frac{H}{t_min} = \frac{Hu_{y,max}}{L} = \frac{1\text{m} \cdot 0.005\frac{\text{m}}{\text{s}}}{1.5\text{m}} = 0.00333\frac{\text{m}}{\text{s}}$$

(e) For Stokes flow, the relationship was derived in Problem Set 8:

$$V = \frac{d^2 g|\rho_s - \rho|}{18\eta} \Rightarrow d = \sqrt{\frac{18\eta V}{g|\rho_s - \rho|}} = \sqrt{\frac{18 \cdot 5.2 \times 10^{-3}\,\frac{\text{kg}}{\text{m}\cdot\text{s}} \cdot 0.00333\,\frac{\text{m}}{\text{s}}}{9.8\,\frac{\text{m}}{\text{s}^2} \cdot (7000\,\frac{\text{kg}}{\text{m}^3} - 3700\,\frac{\text{kg}}{\text{m}^3})}} = 9.8 \times 10^{-5}\text{m}.$$

Okay, now check Stokes flow:

$$\text{Re} = \frac{\rho_{fl} U_\infty d}{\eta} = \frac{7000\,\frac{\text{kg}}{\text{m}^3} \cdot 0.00333\,\frac{\text{m}}{\text{s}} \cdot 9.8 \times 10^{-5}\text{m}}{5.2 \times 10^{-3}\,\frac{kg}{m\cdot s}} = 0.44.$$

Stokes flow is not strictly valid, but the Stokes flow drag force expression is accurate up to a Reynolds number of 1, so this is okay.

So this tundish removes all ceramic particles of this density down to a diameter of about 100 microns, which is quite good!

(f) With a Reynolds number of 673, the entrance length L_e goes as:

$$\frac{L_e}{H} = \frac{\text{Re}_H}{100},$$

so the entrance length is about 6.7 times the 0.15m channel width, or about a meter. Flow is therefore not likely to be fully-developed, since this is a large fraction of the 1.5m channel length.

The real bad news of this is that uneven flow characteristics at the entrance will persist throughout much or all of the channels, and high-velocity regions will flow through quickly and "short circuit" the attempt to float out the ceramic particles. It's not clear at this point whether this difficulty can be overcome so the tundish design will succeed.

4. Drag force on a flat plate

(a) Drag force is proportional to the width and to the square root of the length, so it is lower for the longer length and narrower width, *i.e.* 1m edges parallel to the wind.

(b) First we need to know whether flow is laminar or otherwise, which comes from the Reynolds number:

$$\text{Re}_L = \frac{\rho U_\infty L}{\mu} = \frac{1.9\,\frac{\text{kg}}{\text{m}^3} \cdot 0.01\,\frac{\text{m}}{\text{s}} \cdot 1\text{m}}{10^{-5}\,\frac{\text{N}\cdot\text{s}}{\text{m}^2}} = 1.9 \times 10^3$$

Since this is below 10^5, laminar sounds good. Also, the ratio of boundary layer thickness to length $\frac{\delta}{L}$ is about 0.11, so the $x >> \delta$ assumption holds and the boundary layer analysis with it. Whether 11 cm is much less than 25 cm (so we can neglect edge effects) is possibly arguable, but we'll accept it for an estimate.

Now we have the dimensionless correlation for drag force due to laminar flow past a flat plate:

$$f_L = \frac{1.328}{\sqrt{\text{Re}_L}} = 0.0305$$

$$F_d = fKA = f \times \frac{1}{2}\rho U_\infty^2 \times LW$$

$$F_d = 0.0304 \times \frac{1}{2} \cdot 1.9\frac{\text{kg}}{\text{m}^3} \cdot \left(0.01\frac{\text{m}}{\text{s}}\right)^2 \times 1\text{m} \cdot 0.25\text{m} = 7.24 \times 10^{-7}\text{N}$$

This is the force on one side of the plate; the total force is twice this, about 1.5μN. That's pretty small!

(c) The wind velocity is 1000 times larger than before, so Re_L is 1.9×10^6, and flow is transitional over much of the plate. The ratio $\frac{\delta}{L}$ (using the turbulent flow correlation) is now about 0.021, so we have a thinner boundary layer than before. (If flow were still laminar, it would be about 30 times thinner than in part 4b due to the larger velocity, so the turbulence effectively makes it about 10 times thicker than a laminar boundary layer would be.)

To calculate the friction factor, one can use the graph presented in the notes, or a correlation function. For this Reynolds number, the graph gives a friction factor of about 3.8×10^{-3}, giving a drag force of about 0.09N.

There are two different correlation functions for the drag force. First the one from P&G[1] p. 83:

$$f = \frac{0.455}{(\log \mathrm{Re}_L)^{2.58}} = \frac{0.455}{6.28^{2.58}} = 4.0 \times 10^{-3}$$

And from BSL[2] p. 203:

$$f = \frac{0.146}{\mathrm{Re}_L^{0.2}} = 8.1 \times 10^{-3}$$

So the two friction factors are a factor of two apart! The one in P&G is closer to the graph, but that could be because the graph is in P&G...

The force is calculated using the same $F_d = fKA$ equation as in part 4b, and we get

$$F_d = 4.0 \times 10^{-3} \times \frac{1}{2} \cdot 1.9\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \times 1\mathrm{m} \cdot 0.25\mathrm{m} = 0.095\mathrm{N}$$

(or about twice this using the BSL friction factor correlation); again double this for both sides of the plate. This is still pretty small.

[1]D.R. Poirer and G.H.Geiger, *Transport Phenomena in Materials Processing,* Pittsburgh: TMS, 1994.
[2]Bird, Stewart and Lightfoot, *Transport Phenomena.*