March 28, 2005: Friction factors, boundary layers

Tubes: recall laminar velocity profile

$$u_z = \frac{P_1 - P_2}{4\mu L}(R^2 - r^2), \tag{4.42}$$

shear stress at $r = R$:

$$\tau_{rz} = -\mu \frac{\partial u_z}{\partial r} = \frac{P_1 - P_2}{2L} R \tag{4.43}$$

Average velocity is half maximum:

$$u_{av} = \frac{P_1 - P_2}{8\mu L} R^2, \tag{4.44}$$

so the shear can be given in terms of the average velocity:

$$\tau_{rz} = \frac{4\mu u_{av}}{R} = \frac{8\mu u_a v}{d}. \tag{4.45}$$

Laminar flow friction factor:

$$f = \frac{\tau}{\frac{1}{2}\rho V^2} = \frac{\frac{8\mu}{d} u_{av}}{\frac{1}{2}\rho V^2} = \frac{16\mu}{\rho V d} = \frac{16}{\mathrm{Re}}. \tag{4.46}$$

To calculate drag force: Reynolds number (and surface roughness) $\rightarrow$ friction factor $f \rightarrow \tau = fK$, $F_d = fKA$. Difference with sphere: no stagnation, sudden transition to turbulence.

"Boundary layers" in a solid Thought experiment with moving solid: extruded polymer thick plate (like PS2 extruded rod problem). Start at high temp, if thick and well-cooled so large Biot then constant temperature on surface; no generation. Full equation:

$$\frac{DT}{Dt} = \alpha \nabla^2 T. \tag{4.47}$$

Here, $u_y = 0$, and for $\delta_T << x$ so boundary layer is thin, then

$$\frac{\partial^2 T}{\partial y^2} >> \frac{\partial^2 T}{\partial x^2}, \tag{4.48}$$

can simplify to:

$$u_x \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{4.49}$$

Transform: time $t = x/u_x$, becomes diffusion equation, erf solution:

$$\frac{T - T_s}{T_i - T_s} = \mathrm{erf} \frac{y}{2\sqrt{\alpha x / u_x}} \tag{4.50}$$

If we define δ as where we get to 0.99, then $\mathrm{erf}^{-1}(.99) = 1.8$, and

$$y = \delta \text{ where } \frac{\delta}{2\sqrt{\alpha x / u_x}} = 1.8 \Rightarrow \delta = 3.6\sqrt{\frac{\alpha x}{u_x}}. \tag{4.51}$$

Obviously breaks down at start $x = 0$, but otherwise sound.

Boundary layers in a fluid Now we want to calculate drag force for flow parallel to the plate.

Similar constant IC to solid, infinite BC, call it U_∞. Difference: BC at $y = 0$: $u_x = u_y = 0$. Have to solve 2-D incompressible steady-state Navier-Stokes:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{4.52}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\rho \partial x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial x^2} \right) \tag{4.53}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\rho \partial y} + \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial x^2} \right). \tag{4.54}$$

Then a miracle occurs, the Blassius solution for $\delta \ll x$ is a graph of u_x/U_∞ vs. $\beta = y\sqrt{U_\infty/\nu x}$; hits 0.99 at ordinate of 5:

$$\delta = 5.0\sqrt{\frac{\nu x}{U_\infty}}. \tag{4.55}$$

Why 5.0, not 3.6? Because there must be vertical velocity due to mass conservation (show using differential mass equation and integral box), carries low-x-velocity fluid upward.

Entrance Length For channel flow between two parallel plates spaced apart a distance H, we can define the entrance length L_e as the point where the boundary layers from each side meet in the middle. The twin Blassius functions are close enough to the parabolic profile that we can say it's fully-developed at that point. So we can plug in the boundary layer equation if flow is laminar:

$$x = L_e \Rightarrow \frac{H}{2} = \delta = 5.0\sqrt{\frac{\nu x}{U_a v}}, \tag{4.56}$$

$$L_e = \frac{H^2 U_a v}{100\nu}. \tag{4.57}$$

If $L_e \ll L$, then flow is fully-developed for most of the tube, so the fully-developed part will dominate the drag force and $F_d = \tau \cdot 2\pi R L$.