1. Investment casting heat transfer

   (a) Using the grey body approximation with mold surface emissivity ϵ, the radiative heat flux from the mold surface to the surroundings is given by:

   $$q = \epsilon\sigma T_2^4,$$

   where T_2 is the mold surface temperature. The radiative flux back from the (black, $\epsilon_{env} = 1$) environment to the mold surface is:

   $$q = \sigma T_3^4,$$

   where T_3 is the environment temperature. Since emissivity equals absorbtivity (in the grey body approximation), the amount of heat absorbed by the mold becomes:

   $$q = \epsilon\sigma T_3^4.$$

   The net radiative heat transferred is thus:

   $$q_{net} = \epsilon\sigma(T_2^4 - T_3^4).$$

   (b) Let T_1 be the temperature inside the mold (1450 K), T_2 the outside mold temperature (unknown), and T_3 the temperature of the large room (300 K), and with no information given, assume the large room is a black surface. The conductive flux through the mold is

   $$q = k\frac{T_1 - T_2}{L},$$

   where L is the mold thickness given as 10 mm. If we add the convective flux to the net radiative flux from part 1a, the total heat flux away from the mold surface is:

   $$q = \epsilon\sigma(T_2^4 - T_3^4) + h(T_2 - T_3).$$

   Set the convective and radiative plus convective fluxes equal and solve for T_2:

   $$k\frac{T_1 - T_2}{L} = \epsilon\sigma(T_2^4 - T_3^4) + h(T_2 - T_3)$$

   $$\epsilon\sigma T_2^4 + \left(\frac{k}{L} + h\right) T_2 - \left(\epsilon_2\sigma T_3^4 + hT_3 + \frac{kT_1}{L}\right) = 0$$

   $$2.268 \times 10^{-8}\frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}^4}T_2^4 + (200 + 100)\frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}}T_2 - (183.7 + 90000 + 260000)\frac{\mathrm{W}}{\mathrm{m}^2} = 0$$

   Call the left side $f(T)$ and use Newton's method

   $$T_{i+1} = T_i - \frac{f(T_i)}{f'(T_i)}$$

   Start with an initial guess halfway between T_1 and T_3, and the iterations go (don't let the subscripts confuse you, this is still T_2):

   $$T_0 = 800\mathrm{K},\ T_1 = 1091\mathrm{K},\ T_2 = 1068.863989\mathrm{K},\ T_3 = 1068.673378\mathrm{K},\ T_4 = 1068.673365\mathrm{K}$$

   This is converging pretty quickly to $T_2 = 1068.7$ K.

   (c) If the environment temperature is zero, then the convective heat flux is hT, so the total flux is:

   $$q = \epsilon\sigma T^4 + hT$$

   Factoring out T gives

   $$q = (\epsilon\sigma T^3 + h)T$$

   So if we want a h_{total} for which $q = h_{\text{total}}T$, that would be $\epsilon\sigma T^3 + h$.

(d) The Biot number is:

$$\text{Bi} = \frac{hL}{k}$$

Using the h_{total} above:

$$\text{Bi} = \frac{(\epsilon\sigma T^3 + h)L}{k}$$

At 800K:

$$\text{Bi} = \frac{\left(0.4 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4} \cdot (800\text{K})^3 + 100 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}\right) \cdot 0.01\text{m}}{2 \frac{\text{W}}{\text{m} \cdot \text{K}}} = 0.56$$

At 1050K:

$$\text{Bi} = \frac{\left(0.4 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4} \cdot (1100\text{K})^3 + 100 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}\right) \cdot 0.01\text{m}}{2 \frac{\text{W}}{\text{m} \cdot \text{K}}} = 0.66$$

At 1700K:

$$\text{Bi} = \frac{\left(0.4 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4} \cdot (1700\text{K})^3 + 100 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}\right) \cdot 0.01\text{m}}{2 \frac{\text{W}}{\text{m} \cdot \text{K}}} = 1.05$$

All of these are in the mixed conduction/convection limited regime, which means temperature is somewhat non-uniform through the mold. Recall that for steady-state (heat) diffusion, a Biot number of 1 means that the convection and conduction resistances are about the same, so the surface temperature is halfway between the mold interior temperature (the metal temperature) and the environment temperature; this is quite non-uniform for these purposes.

(e) The temperature calculated in part 1b is close to the 1100K second case in part 1d. The Biot number is equal to the ratio of temperature differences, which from part 1b gives:

$$\text{Bi} = \frac{T_1 - T_2}{T_2 - T_3} = \frac{1300 - 1068.7}{1068.7 - 300} = 0.30,$$

so the results are somewhat different. This is because the Biot number analysis underestimates the flux from the environment back to the mold.

However, qualitatively, this is still in the mixed convection/conduction limited regime, so the temperature is somewhat nonuniform.

(f) We want to lower the Biot number to make this more uniform. We can't reduce L by making the mold thinner; if we try to make it stronger by making it thicker, then the temperature difference across it will only increase and make failure more likely. We may be able to choose a different mold material with higher thermal conductivity.

But the most effective thing to do is to try to reduce h_{total}, which is done in industry by wrapping the mold in a ceramic fiber "blanket", which reflects much of the radiant heat back to the mold and creates a pocket of stagnant air to reduce the convective losses.