3.044 Test 1

Heat Transfer, Engineering Economics

Solutions

1. Write your name on all of your answer booklets

   There were a lot of different answers to this, but everyone got it right.

2. Heat transfer and injection molding

   (a) The Biot number is given by Bi=hL/k. If you use the full thickness, you get:

   $$L = 1\text{cm} \Rightarrow \text{Bi} = \frac{880\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.01\text{m}}{2.2\frac{\text{W}}{\text{m}\cdot\text{K}}} = 4.$$

   $$L = 2\text{cm} \Rightarrow \text{Bi} = \frac{880\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.02\text{m}}{2.2\frac{\text{W}}{\text{m}\cdot\text{K}}} = 8.$$

   With intermediate Biot numbers, we can't use Newtonian cooling or error functions (nor Fourier series, but we didn't cover those this year), so we need to use the centerline temperature curves on the equation sheet. ("Finite differences" is a technically correct methodology too, but that's hard to do on a test.)

   However, these Biot numbers then *cannot* be used in the graphs, which use half the thickness for the Biot number; on the graphs we use Bi=2 and 4 for 1 cm and 2 cm respectively.

   (b) To calculate the cooling time, first calculate the Fourier number from the dimensionless temperature and Biot number using the graph.

   $$\frac{T - T_{fl}}{T_i - T_{fl}} = \frac{45 - 25}{225 - 25} = 0.1$$

   For the 1 cm thick part, Bi = 2, so the graph says Fo=2.0. Keeping in mind $L = 0.005$ m here (half the thickness), we calculate:

   $$\text{Fo} = 2.0 = \frac{\alpha t}{L^2} \Rightarrow t = 2.0\frac{L^2}{\alpha} = 2.0\frac{L^2 \rho c_p}{k} = 2.0\frac{(0.005\text{m})^2 \cdot 1100\frac{\text{kg}}{\text{m}^3} \cdot 3100\frac{\text{J}}{\text{kg}\cdot\text{K}}}{2.2\frac{\text{W}}{\text{m}\cdot\text{K}}} = 77.5 \text{ seconds.}$$

   For the 2 cm thick part, Bi = 4, so the graph says Fo$\simeq$ 1.5. $L = 0.01$ m, so we calculate:

   $$\text{Fo} = 1.5 = \frac{\alpha t}{L^2} \Rightarrow t = 1.5\frac{L^2}{\alpha} = 1.5\frac{L^2 \rho c_p}{k} = 1.5\frac{(0.01\text{m})^2 \cdot 1100\frac{\text{kg}}{\text{m}^3} \cdot 3100\frac{\text{J}}{\text{kg}\cdot\text{K}}}{2.2\frac{\text{W}}{\text{m}\cdot\text{K}}} = 232 \text{ seconds.}$$

   (c) The cooling time increases by a factor of three when the thickness doubles. A scaling law would say that $t \propto L^n$ for some n which we can calculate from these two data points using:

   $$n = \frac{\log(t_2/t_1)}{\log(L_2/L_1)} = \frac{\log(3)}{\log(2)} = 1.58.$$

This makes sense: if conduction-limited, time scales as thickness squared; if limited by the mold and/or fluid boundary layer (captured together in h), time scales linearly with thickness; this is in between.

3. Cost, scale and process selection

   (a) Note: "RTM units" in the problem meant machines, but due to ambiguous wording, either this or effective production volume were acceptable answers.

$$\#\text{ of RTM Units} = \frac{\text{Annual Required Operating Time}}{\text{Annual Available Operating Time}}$$

$$\text{Required Operating Time} = PV_{eff} \times \text{Cycle Time}$$

$$PV_{eff} = \frac{\text{Production Volume}}{\text{Yield}} = \frac{5,000}{0.85} = 5,883$$

$$\text{Required Operating Time} = 5,883 \times \frac{1,800\text{sec}}{3,600\text{sec/hr}} = 2,941\text{hours}$$

$$\#\text{ of RTM Units} = \frac{2,941\text{hours}}{2,500\text{hours}} = 1.18 = 2\text{ machines}$$

   (b) Calculate total unit cost:

$$\text{Total Unit Cost} = \frac{\text{Annual Equivalent Cost}}{\text{Annual Production Volume}}$$

$$\text{Annual Production Volume} = 5,000$$

$$\text{Annual Equivalent Cost} = \text{Annual Equivalent Capital Cost} + \text{Annual Materials Cost}$$

$$\begin{aligned}
\text{Annual Materials Cost} &= \text{Annual Usage} \times \text{Unit Price} \\
&= (PV_{eff} \times \text{Part Mass}) \times \text{Unit Price} \\
&= 5,883 \times 2 \times \$4 \\
&= \$47,059 \\
\text{Annual Equivalent Capital Cost} &= \text{Investment} \times r\frac{(1+r)^N}{(1+r)^N - 1} \\
&= (\#\text{ of Equip} \times \text{Equip Price}) \times 0.15\frac{(1.15)^5}{(1.15)^5 - 1} \\
&= 2 \times \$1\text{million} \times 0.298 \\
&= \$596,600 \\
\text{Annual Equivalent Cost} &= \$596,600 + \$47,060 = \$643,700 \\
\text{Unit Cost} &= \frac{\$643,700}{5,000} = \$128.74
\end{aligned}$$

   (c) First lets compare costs when each process is operating one machine. For RTM this would allow us to produce up to 4250 parts per year, the injection process would allow 19,615.

$$\begin{aligned}
\text{Unit Cost}_{\text{RTM}} &= \text{Unit Cost}_{\text{PolyCorp}} \\
\frac{\$1\text{million} \times 0.298 + (2\text{kg} \times \$4/\text{kg} \times PV_{eff})}{PV} &= \frac{\$1.8\text{million} \times 0.298 + (2\text{kg} \times \$2/\text{kg} \times PV_{eff})}{PV} \\
\frac{\$298,300}{PV} + \frac{\$8}{85\%} &= \frac{\$537,000}{PV} + \frac{\$4}{85\%} \\
PV &= 50,713
\end{aligned}$$

But this answer is much larger than the production capacity for one machine of either process. So let's try two RTM machines—basically, looking at the production volume range of 4,251 to 8,500.

$$
\begin{aligned}
\text{Unit Cost}_{\text{RTM}} &= \text{Unit Cost}_{\text{PolyCorp}} \\
\frac{\$2\text{million} \times 0.298 + (2\text{kg} \times \$4/\text{kg} \times PV_{eff})}{PV} &= \frac{\$1.8\text{million} \times 0.298 + (2\text{kg} \times \$2/\text{kg} \times PV_{eff})}{PV} \\
\frac{\$596,600}{PV} + \frac{\$8}{85\%} &= \frac{\$537,000}{PV} + \frac{\$4}{85\%} \\
PV &= -12,665
\end{aligned}
$$

There is no positive PV which will make the RTM option cheaper with two machines.

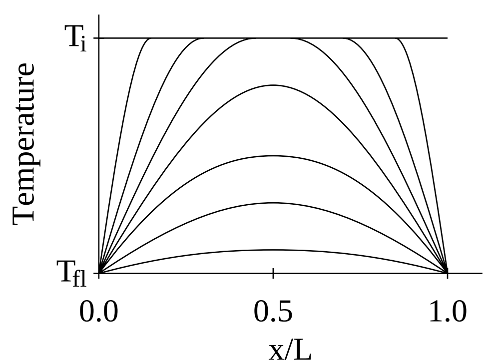
Therefore, PolyCorp's process is cheaper as soon as production volume exceeds the level requiring two RTM machines, which is 4,250 units/year.

4. Crystal-free zone in a glass-ceramic dish

(a) The Biot number is:

$$
\text{Bi} = \frac{hL}{k} = \frac{3000\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.01\text{m}}{0.4\frac{\text{W}}{\text{m}\cdot\text{K}}} = 75.
$$

Since this is very large (even using half the thickness for the lengthscale), assume the surface cools to T_{fl} very rapidly (and assume this material somehow survives the resulting thermal stresses). Taking x to be the distance from one side, temperature sketches should look like:



(b) With a uniform initial condition, and a constant T boundary condition, at short time scales the error function is the appropriate solution to the thermal diffusion equation. Since the surface temperature (equal to the fluid temperature) is lower than the initial temperature, the erf is easier to use than the erfc:

$$
\frac{T - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right).
$$

(c) We want to solve for x where $T = T_{nose}$ at time $t = t_{nose}$. The relative temperature at T_{nose} is:

$$
\frac{T_{nose} - T_{fl}}{T_i - T_{fl}} = \frac{720 - 300}{1000 - 300} = 0.6.
$$

From the error function table, $\text{erf}^{-1}(0.6) \simeq 0.6$, so we need to set the argument of the erf to 0.6:

$$
\frac{x}{2\sqrt{\alpha t}} = 0.6 \Rightarrow x = 1.2\sqrt{\alpha t} = 1.2\sqrt{\frac{0.4\frac{\text{W}}{\text{m}\cdot\text{K}}}{2400\frac{\text{kg}}{\text{m}^3} \cdot 900\frac{\text{J}}{\text{kg}\cdot\text{K}}}4\text{seconds}} = 0.00103\text{m}.
$$

[Using the erfc solution, the dimensionless temperature is 0.4, and $\text{erfc}^{-1}(0.4) \simeq 0.6$, resulting in the same answer.]

So the all-glassy crystal-free zone is about a millimeter thick on each side.

(d) The criterion for validity of the error function is:

$$\frac{L}{2\sqrt{\alpha t}} \geq 2,$$

$$\frac{L^2}{16\alpha} \geq t.$$

Here L should be *half of the thickness* since when they meet in the middle, the error function is no longer valid. This gives $t \leq 8.73$ seconds, and since 4 seconds is less than this, the error function is still valid.

Alternatively, at four seconds, this gives $L \geq 0.0034$m, and since the half-thickness is 0.005m, the validity criterion is satisfied. Or, at $L = 0.005$m and $t = 4$ seconds, $L/2\sqrt{\alpha t} = 2.9$ which is more than two, with the same conclusion.

5. Dimensional analysis: mixed radiation and conduction

(a) Parameters here are: the heat flux q, shell thickness L, shell thermal conductivity k, shell surface emissivity times radiation constant $\epsilon\sigma$ (since the problem asked you to group them), and inner surface temperature T_{in}. This makes:

$$q = f(L, k, \epsilon\sigma, T_{in}).$$

Note that convective cooling is not mentioned anywhere, so the heat transfer coefficient h should not be part of this analysis. Also, since ρ and c_p have to do with heat required to change temperature, these are not relevant to a steady-state problem.

(b) The units for these five parameters are, respectively: $\frac{\text{W}}{\text{m}^2}$, m, $\frac{\text{W}}{\text{m}\cdot\text{K}}$, $\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}$, and K. The minimal set of units needed to derive all of these are W, m and K, three base units.

With five parameters and three base units, the Buckingham pi theorem tells us we should have two dimensionless parameters.

(c) As is common to dimensional analysis, this part allows multiple possible right answers. There are three good choices here, and one very bad choice. Good choices are:

i. Keep q and k and eliminate $\epsilon\sigma$, L and T_{in}:

$$\pi_q = \frac{q}{\epsilon\sigma T_{in}^4}, \quad \pi_k = \frac{k}{\epsilon\sigma T_{in}^3 L}.$$

ii. Keep q and L and eliminate $\epsilon\sigma$, k and T_{in}:

$$\pi_q = \frac{q}{\epsilon\sigma T_{in}^4}, \quad \pi_L = \frac{\epsilon\sigma T_{in}^3 L}{k}.$$

iii. Keep q and $\epsilon\sigma$ and eliminate L, k and T_{in}:

$$\pi_q = \frac{qL}{kT_{in}}, \quad \pi_{\epsilon\sigma} = \frac{\epsilon\sigma T_{in}^3 L}{k}.$$

For these choices, dimensionless flux is the ratio of actual flux to what the flux would be if only radiation (first two) or conduction (third) dominated the resistance respectively. The other dimensionless parameter is a sort of "radiation Biot number" (or its inverse in the first choice) with $\epsilon\sigma T_{in}^3$ playing the role of heat transfer coefficient, indicating the relative resistances due to conduction and radiation.

The poor choice is:

iii. Keep q and T_{in} and eliminate $\epsilon\sigma$, k and L:

$$\pi_q = \frac{q(\epsilon\sigma)^{\frac{1}{3}} L^{\frac{4}{3}}}{k^{\frac{4}{3}}},\ \pi_{T_{in}} = \frac{T_{in}(\epsilon\sigma)^{\frac{1}{3}} L^{\frac{1}{3}}}{k^{\frac{1}{3}}}.$$

Here, $\pi_{T_{in}}$ is the cube root of the "radiation Biot number" above, and π_q is the ratio qL/kT_{in} times $\pi_{T_{in}}$.

(d) When the "radiation Biot number" is small (or its inverse π_k is large in the first choice), the radiation resistance is large, so that becomes the limiting factor. Another way to look at this is that a small Biot number means a uniform temperature in the solid. In this case, the temperature of the outer surface of the shell is approximately T_{in}, so the flux is $q = \epsilon\sigma T_{in}^4$.

When this number is large (or π_k is small), the conduction resistance dominates, so the surface temperature is the environment temperature (approximately zero relative to T_{in}, and flux is $q = kT_{in}/L$.

This gives scaling behavior for the "three good choices" as follows:

i. π_q and π_k: When π_k (inverse Biot) is large, π_q is 1 because it is the ratio of q to $\epsilon\sigma T^4$. When π_k is small, conduction resistance reduces flux below the radiative flux, so q falls below $\epsilon\sigma T^4$, and in the limit of small π_k, falls to zero.

ii. π_q and π_L: Since π_L is the opposite of π_k above, when it is close to zero, π_q goes to 1; when π_L gets very large, π_q goes to zero.

iii. π_q and $\pi_{\epsilon\sigma}$: Here π_q is the ratio of actual flux to conduction-limited flux. When $\pi_{\epsilon\sigma}$ is large, flux is conduction-limited, and π_q goes to 1. When it is small, radiation adds significant limitation, so flux is much smaller than the conduction-limited flux and π_q goes to zero.

iv. π_q and $\pi_{T_{in}}$: as mentioned, $\pi_{T_{in}}$ is the cube root of the "radiation Biot number". When it is small, radiation dominates resistance, so qL/kT_{in} gets small as in the previous case, making both parts of π_q go to zero. When it is large, flux is conduction-limited, so qL/kT_{in} goes to one, and the product scales as the dimensionless temperature:

$$\pi_q = \frac{qL}{kT_{in}} \frac{T_{in}(\epsilon\sigma)^{\frac{1}{3}} L^{\frac{1}{3}}}{k^{\frac{1}{3}}} \longrightarrow 1 \cdot \pi_{T_{in}}.$$

This is by far the most complicated of the three, in terms of the exponents, connection to the physics, and asymptotic behavior, making this the "poor" choice of parameters.