

### 3.044 Recitation 6

March 17-18, 2005

#### Topics

- Fluid Flow & Drag Force on a sphere
- Examples: Flow over a moving inclined plane, Upward Flow in a tube
- PSET 4: Questions

#### Velocity Profile Derivation

Momentum Balance = Pressure Gradient + Shear Force (Viscous Force) + Gravity Force

	Two Parallel Plate	Inclined Plane	Horizontal Tube
Governing Eqn	$\frac{\partial(\rho u_x)}{\partial t} = -\frac{\partial P}{\partial x} - \frac{\partial \tau_{yx}}{\partial y}$	$\frac{\partial(\rho u_x)}{\partial t} = -\frac{\partial \tau_{yx}}{\partial y} + \rho g \sin \theta$	$\frac{\partial(r\rho u_z)}{\partial t} = -\frac{\partial P}{\partial z} r - \frac{\partial}{\partial r}(r\tau_{rz})$
$\tau_{yx} = -\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$ General Solution Newtonian, Incompressible	$u_x = -\frac{P_1 - P_2}{2\mu L} y^2 + Ay + B$	$u_x = -\frac{\rho g \sin \theta}{2\mu} y^2 + Ay + B$	$u_z = -\frac{P_1 - P_2}{4\mu L} r^2 + A \ln r + B$
B.C.	At $y = \delta, u_x = 0$ (non slip) At $y=0, \tau_{yx} = 0$ (centerline)	At $y=L, u_x = 0$ At $y=0, \tau_{yx} = 0$ (Free surface)	At $r=R, u_z = 0$ At $r=0, \frac{\partial u_z}{\partial r} = 0$ (symmetric flow)
Steady State $u$ $\frac{\partial(\rho u)}{\partial t} = 0$	$u_x(y) = \frac{P_1 - P_2}{2\mu L} (\delta^2 - y^2)$	$u_x(y) = \frac{\rho g \sin \theta}{2\mu} (L^2 - y^2)$	$u_z(r) = \frac{P_1 - P_2}{4\mu L} (R^2 - r^2)$
$u_{max} @ \frac{\partial u}{\partial n} = 0$	$u_{max} = \frac{P_1 - P_2}{2\mu L} \delta^2$	$u_{max} = \frac{\rho g \sin \theta}{2\mu} L^2$	$u_{max} = \frac{P_1 - P_2}{4\mu L} R^2$
$u_{avg} = \frac{\int \vec{u} \cdot \hat{n} dA}{area}$	$\frac{\int_0^\delta u_x dy}{\delta} = \frac{P_1 - P_2}{3\mu L} \delta^2$	$\frac{\int_0^L u_x dy}{L} = \frac{\rho g \sin \theta L^2}{3\mu}$	$\frac{\int_0^R u_z 2\pi r dr}{\pi R^2} = \frac{P_1 - P_2}{8\mu L} R^2$
Flow rate $Q = u_{avg} area$	$Q = \frac{2(P_1 - P_2)}{3\mu L} W \delta^3$	$Q = \frac{\rho g \sin \theta}{3\mu} W L^3$	$Q = \frac{\pi(P_1 - P_2)}{8\mu L} R^4$

**Drag Force for a sphere**  $F_d = (\text{Friction Factor}) (K)(\text{Area}) = f\left(\frac{1}{2}\rho V^2\right)\left(\frac{1}{4}\pi d^2\right) = 3\pi d\mu V$  (Stokes Flow:  $Re < 0.1$ )

$$f = \frac{3\pi d\mu V}{\frac{1}{2}\rho V^2\left(\frac{1}{4}\pi d^2\right)} = \frac{24\mu}{\rho V d} = \frac{24}{Re}$$

A particle travels through a fluid at the terminal velocity when the body weight ( $F_w$ ) is balanced with the sum of the buoyant force ( $F_b$ ) and the drag force ( $F_d$ ).

$$F_w = F_b + F_d$$

$$m_s g = m_{fluid} g + 3\pi d \mu V_t$$

$$\frac{1}{6} \pi d^3 \rho_s g = \frac{1}{6} \pi d^3 \rho_{fluid} g + 3\pi d \mu V_t$$

**Example 1.** Flow on an inclined plane which is moving at velocity  $u_0$ . Derive the velocity profile.

(a) List all assumptions to solve for a steady state solution

- Newtonian Fluid  $\tau_{yx} = -\mu(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})$
- Fully developed flow: at some distance far from the beginning of the flow  $u_x = f(y)$
- Incompressible fluid: uniform density

(b) General Solution: Since all of the above assumptions are the same as the inclined plane case, we can use the general solution give in the table above

$$u_x = -\frac{\rho g \sin \theta}{2\mu} y^2 + Ay + B$$

(c) Apply the following boundary conditions. \*Note: In this example, the reference axis is different from what given in the table above to show that there can be more than one forms of solution to the velocity profile depending on the frame of reference.\*

$$\text{B.C. 1 } y=L, \frac{\partial u_x}{\partial y} = 0 \text{ (Free surface)}$$

$$0 = -\frac{\rho \sin \theta}{2\mu} 2y|_{y=L} + A$$

$$A = \frac{\rho \sin \theta}{\mu} L$$

$$\text{B.C. 2 } y=0, u_x = u_0$$

$$u_0 = -\frac{\rho g \sin \theta}{2\mu} (0)^2 + \frac{\rho \sin \theta}{\mu} L(0) + B$$

$$u_x(y) = \frac{\rho \sin \theta}{2\mu} (2Ly - y^2) + u_0$$

$$\tau_{yx} = -\frac{\mu \partial u_x}{\partial y} = -\frac{\rho \sin \theta}{2} (2L - 2y)$$

\* If there are multilayers fluids, at the interface the shear forces have to be equal (analogous to heat transfer problem where heat flux is equal.)

(d) Calculate  $u_{max}$

$$u_{max} \quad @ \quad y = L, u_{max} = \frac{\rho \sin \theta}{2\mu} L^2 + u_0$$

(e) Calculate  $u_{avg}$

$$u_{avg} = \frac{1}{L} \int_0^L [\frac{\rho \sin \theta}{2\mu} (2Ly - y^2) + u_0] dy = \frac{1}{L} [\frac{\rho \sin \theta}{2\mu} (2L \frac{y^2}{2} - \frac{y^3}{3}) + u_0 y]_0^L = \frac{\rho g \sin \theta L^2}{3\mu} + u_0$$

**Example 2.** Derive the velocity equation for an upward flow in a tube.

(a) Assumptions

- Newtonian Fluid  $\tau_{rz} = -\mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$
- Fully developed flow: at some distance far from the beginning of the flow  $u_z = f(r)$
- Incompressible fluid: uniform density

(b) Governing Equation

$$\frac{\partial(r\rho u_z)}{\partial t} = -\frac{\partial P}{\partial z}r - \frac{\partial}{\partial r}(r\tau_{rz}) + \rho gr$$

(c) Derive a general solution

$$\begin{aligned} 0 &= \frac{P_1 - P_2}{L}r + \frac{\partial}{\partial r}(r\mu \frac{\partial u_z}{\partial r}) + \rho gr \\ \frac{\partial}{\partial r}(r\mu \frac{\partial u_z}{\partial r}) &= -\frac{P_1 - P_2}{L}r - \rho gr \\ r\mu \frac{\partial u_z}{\partial r} &= -\frac{P_1 - P_2}{2L}r^2 - \frac{\rho gr^2}{2} + A \\ \frac{\partial u_z}{\partial r} &= -\frac{P_1 - P_2}{2\mu L}r - \frac{\rho gr}{2\mu} + \frac{A}{r} \end{aligned}$$

$$u_z = -\left(\frac{P_1 - P_2}{4\mu L} + \frac{\rho g}{4\mu}\right)r^2 + A \ln r + B$$

(d) What are the boundary conditions? BC1:  $r = 0, \frac{\partial u_z}{\partial r} = 0$ ; BC2:  $r = R, u_z = 0$

(e) Plug the given B.C. into (c) to get the velocity profile for the upward flow in a tube

$$\text{BC1: } r = 0, \frac{\partial u_z}{\partial r} = 0$$

$$\begin{aligned} 0 &= -\frac{P_1 - P_2}{2\mu L}(0) - \frac{\rho g(0)}{2\mu} + \frac{A}{r} \\ A &= 0 \end{aligned}$$

$$\text{BC2: } r = R, u_z = 0$$

$$\begin{aligned} 0 &= -\left(\frac{P_1 - P_2}{4\mu L} + \frac{\rho g}{4\mu}\right)R^2 + B \\ B &= \left(\frac{P_1 - P_2}{4\mu L} + \frac{\rho g}{4\mu}\right)R^2 \end{aligned}$$

$$u_z = \left(\frac{P_1 - P_2}{4\mu L} + \frac{\rho g}{4\mu}\right)(R^2 - r^2)$$