3.044 Problem Set 4

Solidification and Fluids

Solutions

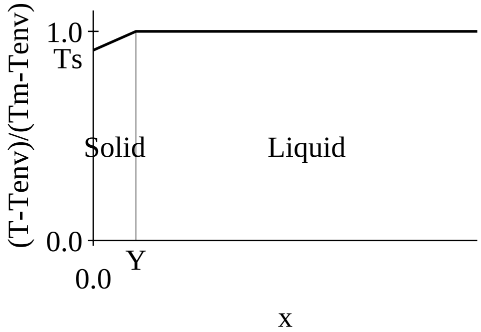
1. Heat conduction and diffusion in alloy casting

   (a) This was a bit confusing because we used a single coefficient h to describe both convective transfer from the mold to the environment, and also conduction through the mold.

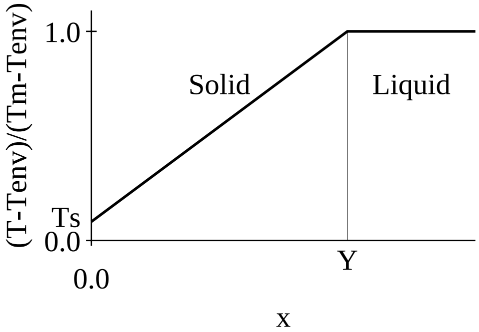
   That said, for pseudo-steady-state conduction, temperature is linear across the solid metal, varying between (unknown) T_s at the mold-metal interface and T_m at the melt interface. The Biot number hY/k determines the ratio of temperature differences:

   $$\frac{hY}{k} = \frac{T_m - T_s}{T_s - T_{env}}.$$

   For a small Biot number, this ratio will be small, putting T_s closer to T_m; for a large Biot number, T_s wil be closer to T_{env}. So your small- and large-Biot number sketches should have looked something like:



Temperature, Bi=0.1



Temperature, Bi=10

   (b) Start with a heat balance at a moving melt interface (from the equation sheet):

   $$|q_\ell| - |q_s| = -U\rho\Delta H_f$$

   where $U = dY/dt$, the velocity of the melt interface. Because the liquid temperature is uniform, there's no temperature gradient there, and no flux, so $q_\ell = 0$. This leaves us with the solid, where $q_s = k_s dT/dx$. But since we don't know $T_m - T_s$ very well (see the left graph above), we can instead use the expression given in the problem, simplified since $T_s \simeq T_m$:

   $$q_s = h(T_s - T_{env}) \simeq h(T_m - T_{env})$$

   This gives us:

   $$\frac{dY}{dt} = U = \frac{h(T_m - T_{env})}{\rho\Delta H_f}$$

   This leads to a roughly constant front velocity, and linear growth of the solid layer.

(c) For long times (large Biot number), growth is limited by conduction through the solid metal, and $T_s \simeq T_{env}$. In this case, we can use the pseudo-steady-state approximation to estimate the temperature gradient to give the heat flux as:

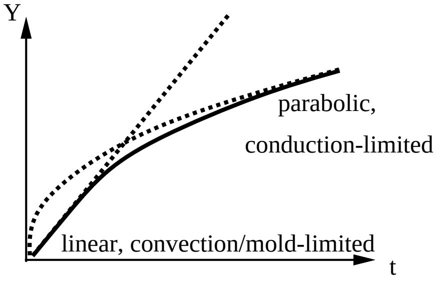
$$q_s = -k_s \frac{\partial T}{\partial x} = -k_s \frac{T_m - T_s}{Y} \simeq -k_s \frac{T_m - T_{env}}{Y}.$$

This too goes into the melt interface heat balance, giving the solidification rate:

$$\frac{dY}{dt} = \frac{k_s(T_m - T_{env})}{Y \rho \Delta H_f}$$

Solving this differential equation gives $Y \propto \sqrt{t}$.

(d) This sketch should have included combined and parabolic growth rates something like:



(e) This part involves solute diffusion in a moving frame of reference, which gives us convective mass transfer. We start with the equation for diffusion in a moving body (from the equation sheet):

$$\frac{\partial C}{\partial t} + u_x \frac{\partial C}{\partial x'} = D \frac{\partial^2 C}{\partial x'^2} + G.$$

This is steady-state, so $\partial C/\partial t = 0$, and in a metal alloy, the species are elements which aren't created or destroyed, so $G = 0$. We're left with an ordinary differential equation:

$$D \frac{d^2 C}{dx'^2} - u_x \frac{dC}{dx'} = 0$$

As a homogeneous equation with constant coefficients, we can solve this by assuming a solution of the form $C = e^{Rx'}$ and solve for R:

$$DR^2 e^{Rx'} - u_x R e^{Rx'} = 0$$

$$R(DR - u_x) = 0$$

$$R = 0 \quad \text{or} \quad R = \frac{u_x}{D}$$

$$C = A \exp\left(\frac{u_x}{D} x'\right) + B$$

(f) In the liquid, the concentration at the interface where $x' = 0$ is $5C_L$; a long way from the interface (at $x' = \infty$), $C = C_L$. Taking the second condition first, and noting that u_x is negative (since the interface is moving in the positive x-direction):

$$C_L = Ae^{-\infty} + B \;\Rightarrow\; B = C_L,$$

then the first condition:

$$5C_L = Ae^0 + C_L \;\Rightarrow\; A = 4C_L;$$

$$C = 4C_L \exp\left(\frac{u_x}{D} x'\right) + C_L.$$

(g) The order-of-magnitude thickness of the high-concentration layer ends where the argument of the exponential is -1 (for simplicity):

$$\frac{u_x}{D}x' = -1 \;\Rightarrow\; x' = -\frac{D}{u_x}.$$

To get the thickness to a specific tolerance, *e.g.* the 1% criterion like a boundary layer, we just multiply this by a constant such as ln(0.01).

2. Freezing by radiation and convection

(a) The thermal conductivity is given by the Wiedmann-Franz law:

$$k_{el} = L\sigma_{el}T = 2.45 \times 10^{-8}\frac{\mathrm{W\Omega}}{\mathrm{K}^2} \cdot 5 \times 10^5(\Omega \cdot \mathrm{m})^{-1} \cdot 1800\mathrm{K} = 22\frac{\mathrm{W}}{\mathrm{m} \cdot \mathrm{K}}.$$

(b) Total flux away from the top surface is radiative plus convective (T_s is top surface temperature):

$$q_{total} = q_{rad} + q_{conv} = \epsilon\alpha_{env}\sigma(T_s^4 - T_{env}^4) + h(T_s - T_{env}).$$

(c) If the environment is much colder, then $T_s \gg T_{env}$ so $T_{env} \simeq 0$. If it is "black", then its absorbtivity is one. So the above expression simplifies to

$$q_{total} = \epsilon\sigma T_s^4 + hT_s = h_{total}T_s,$$

$$h_{total} = \epsilon\sigma T_s^3 + h = 200 + 100\frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}} = 300\frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}}.$$

(d) Set the Biot number to 0.1 and solve for Y using h_{total}:

$$\mathrm{Bi} = \frac{h_{total}Y}{k} = 0.1 \Rightarrow Y = \frac{0.1k}{h_{total}} = \frac{0.1 \cdot 22\frac{\mathrm{W}}{\mathrm{m\cdot K}}}{300\frac{\mathrm{W}}{\mathrm{m^2\cdot K}}} = 0.0073\mathrm{m}(7.3\mathrm{mm}).$$

(e) You're given an equation relating flux to solidification rate, so solve it for dY/dt:

$$q_L - q_S = \rho\Delta H_f\frac{dY}{dt} \Rightarrow \frac{dY}{dt} = \frac{q_L - q_S}{\rho\Delta H_f}.$$

Since liquid metal temperature is uniform, $q_L = 0$. At quasi-steady-state, the flux through the solid is equal to the flux leaving its top surface, which is $h_{total}T_s$, T_s being the surface temperature. Since the metal temperature is roughly uniform, $T_s \simeq T_m$ so we can use that:

$$\frac{dY}{dt} = \frac{h_{total}T_m}{\rho\Delta H_f} = \frac{300\frac{\mathrm{W}}{\mathrm{m^2\cdot K}} \cdot 1800\mathrm{K}}{7500\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 2.67 \times 10^5\frac{\mathrm{J}}{\mathrm{kg}}} = 2.7 \times 10^{-4}\frac{\mathrm{m}}{\mathrm{s}}.$$

The time required to reach the thickness calculated above of 7.3mm is that divided by the rate calculated here, or about 27 seconds.

(f) We have resistances in series, due to conduction through the solid and evaporation/convection from the surface. So the heat flux in the solid goes like:

$$q_s = \frac{T_m - T_{env}}{\frac{Y}{k_s} + \frac{1}{h_{total}}}.$$

The trouble is, h_{total} is a function of the surface temperature T_s, which is somewhere between T_{env} and T_m. We can relate T_s to the conductive flux, which is the same as the flux q above:

$$q_s = k_s\frac{T_m - T_s}{Y} \Rightarrow T_s = T_m - \frac{qY}{k_s}.$$

We plug this into the h_{total} expression from part 2c:

$$h_{total} = \epsilon\sigma T_s^3 + h = \epsilon\sigma\left(T_m - \frac{qY}{k_s}\right)^3 + h,$$

then plug that into the heat flux:

$$q_s = \frac{T_m - T_{env}}{\frac{Y}{k_s} + \frac{1}{\epsilon\sigma\left(T_m - \frac{q_s Y}{k_s}\right)^3 + h}}.$$

Then we need to solve this for q_s (probably numerically), and use that in the solidification rate equation from part 2e.

3. Non-Newtonian polymer flow in a channel

   (a) For pseudoplastic fluids, the effective viscosity $-\tau/\dot{\gamma}$ decreases with increasing shear, so n must be less than one. (If one, it's constant; greater than one, it increases with increasing shear.)

   (b) Let's put $y = 0$ halfway between the plates, so the boundary conditions are $y = \pm\delta/2 \rightarrow u_x = 0$. Because we have symmetric boundary conditions and constant (and therefore symmetric) driving force and uniform properties, we can assume the velocity is symmetric about $y = 0$. For this reason, we can say that at $y = 0$, $\frac{du_x}{dy} = 0$, and see how this helps us. We can get $\frac{du_x}{dy}$ from above:

$$\frac{du_x}{dy} = \left(-\frac{\Delta P}{\mu_0 \cdot L} y + C_1\right)^{\frac{1}{n}}$$

and set it to zero at $y = 0$:

$$0 = \left(-\frac{\Delta P}{\mu_0 \cdot L} \cdot 0 + C_1\right)^{\frac{1}{n}}$$

$$0 = C_1^{\frac{1}{n}}$$

So, $C_1 = 0$. What a relief! This simplifies the general solution to:

$$u_x = -\frac{\mu_0 \cdot L}{\Delta P}\frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L} y\right)^{\frac{n+1}{n}} + C_2$$

and we can go a step further to:

$$u_x = \frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}} y^{\frac{n+1}{n}} + C_2$$

This is only strictly valid for non-negative $-\frac{\Delta P}{\mu_0 \cdot L} y$ and therefore for non-positive y. We can get the positive y velocities by symmetry.
Now use the boundary condition $y = -\frac{\delta}{2} \Rightarrow u_x = 0$ and solve for C_2:

$$0 = \frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}}\left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}} + C_2$$

$$C_2 = -\frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}}\left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}}$$

which can also be written

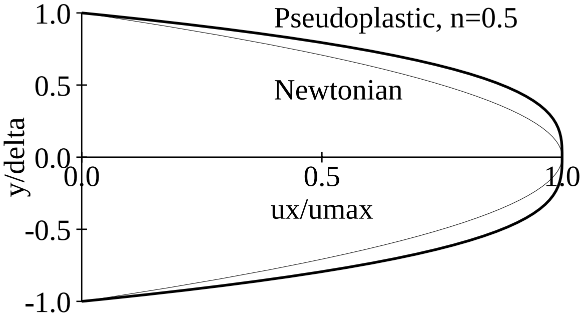
$$C_2 = \frac{n}{n+1}\frac{\delta}{2}\left(\frac{\Delta P}{\mu_0 \cdot L}\frac{\delta}{2}\right)^{\frac{1}{n}}$$

So the velocity profile for non-positive y is given by:

$$u_x = -\frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}}\left[\left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}} - y^{\frac{n+1}{n}}\right]$$

Note that the maximum velocity at $y = 0$, which is given by C_2, is proportional to the thickness to the $1+\frac{1}{n}$ power. For a Newtonian fluid, this increases quadratically in thickness (*i.e.* proportional to thickness squared); for dilatant fluids ($n > 1$), between quadratic and linear; for pseudoplastic fluids ($n < 1$), faster than quadratic. Maximum velocity and flow rate are also nonlinear functions of pressure difference, pseudo-viscosity, and length!

(c) For $n = 0.5$, $u_x \sim y^3$, so this looks like:



(d) The velocity profile is more uniform around the maximum, so the average will be more than 2/3 of the maximum. In fact, for $n = 0.5$, it will be 3/4 of the maximum.

4. Plate glass casting

For this problem, a useful analogy is that of steady-state conduction through a multilayer wall. In that steady-state zero-generation problem, the general solution is $T = Ax + B$, but each layer has a different A and B. Other similarities are described below.

(a) For flow in the x-direction with z normal to the plate, the general solution is:

$$u_x = -\frac{g\sin\theta}{2\nu}z^2 + Az + B.$$

This solution will hold in both the tin and the glass, but with different A and B, as mentioned above.

(b) Just as with heat conduction, where in a multilayer wall the temperatures and normal fluxes are matched at the boundary, here the velocities and shear stresses are equal for the glass and tin at their interface:

$$u_{x,tin} = u_{x,glass};\ \tau_{zx,tin} = \tau_{zx,glass}.$$

(c) Now things start to get ugly. We have two pairs of constants A_{tin}, B_{tin} and A_{glass}, B_{glass} for the two layers. And we have two interface equations (part 4b), and two boundary conditions: zero shear stress at the top of the glass, and zero velocity at the bottom of the tin.

Let's start by setting $z = 0$ at the bottom of the tin and $z = z_1$ at the tin-glass interface. In the glass, we'll use a different coordinate z' measured from the top of the glass, so $z' = 0$ at the top surface, and $z' = z_2$ at the tin-glass interface. Then we can just plug in the boundary conditions:

$$z' = 0 \quad \Rightarrow \quad -\mu_{glass}\left.\frac{\partial u_x}{\partial z'}\right|_{glass} = 0, \tag{1}$$

$$z = z_1, z' = z_2 \quad \Rightarrow \quad u_{x,tin} = u_{x,glass}, \tag{2}$$

$$z = z_1, z' = z_2 \quad \Rightarrow \quad -\mu_{tin} \left.\frac{\partial u_x}{\partial z}\right|_{tin} = \mu_{glass} \left.\frac{\partial u_x}{\partial z'}\right|_{glass}, \tag{3}$$

$$z = 0 \quad \Rightarrow \quad u_{x,tin} = 0. \tag{4}$$

Right away, the first boundary condition at the top indicates that $A_{glass} = 0$, and the last that $B_{tin} = 0$. Since we have $A_{glass} = 0$, and condition 3 deals with A_{tin} and A_{glass}, we can calculate A_{tin} from that boundary condition:

$$\frac{\mu_{tin} g \sin\theta}{\nu_{tin}} z_1 - \mu_{tin} A_{tin} = -\frac{\mu_{glass} g \sin\theta}{\nu_{glass}} z_2,$$

$$A_{tin} = \frac{g \sin\theta}{\mu_{tin}} \left(\rho_{tin} z_1 + \rho_{glass} z_2\right).$$

This is the x-component of the weight per unit area of the glass-tin sandwich, divided by the tin viscosity. Now we need just B_{glass}, which we can get from condition 2 (remembering $B_{tin} = 0$):

$$-\frac{g \sin\theta}{2\nu_{tin}} z_1^2 + A_{tin} z_1 = -\frac{g \sin\theta}{2\nu_{glass}} z_2^2 + B_{glass},$$

$$B_{glass} = \frac{g \sin\theta}{2} \left(\frac{z_2^2}{\nu_{glass}} - \frac{z_1^2}{\nu_{tin}}\right) + \frac{g \sin\theta z_1}{\mu_{tin}} \left(\rho_{tin} z_1 + \rho_{glass} z_2\right),$$

$$B_{glass} = \frac{g \sin\theta}{2} \left(\frac{z_1^2}{\nu_{tin}} + \frac{z_2^2}{\nu_{glass}} + \frac{2 z_1 z_2 \rho_{glass}}{\mu_{tin}}\right).$$

This can be interpreted as the velocity of a plain tin layer, plus that of a plain glass layer, plus a term due to the glass weight driving tin shear; note that if the tin and glass properties were equal, this would simplify to a function of $(z_1 + z_2)^2$. So we have the velocity profile in the glass and tin layers:

$$u_{x,glass} = \frac{g \sin\theta}{2} \left(\frac{z_1^2}{\nu_{tin}} + \frac{z_2^2 - z'^2}{\nu_{glass}} + \frac{2 z_1 z_2 \rho_{glass}}{\mu_{tin}}\right),$$

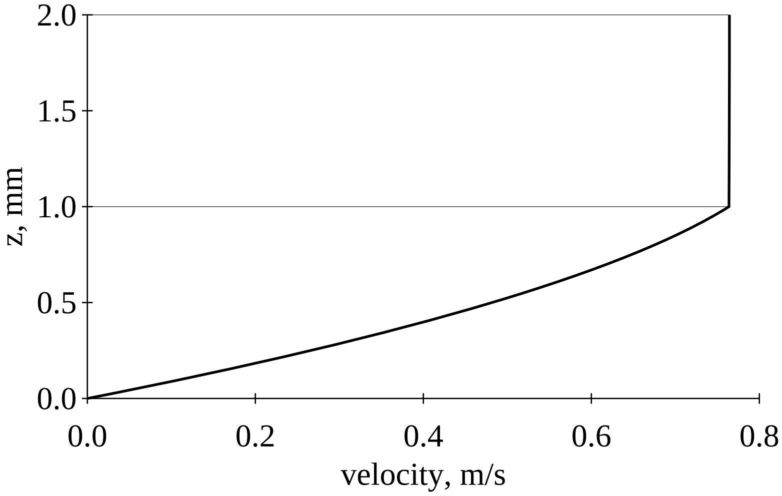
$$u_{x,tin} = \frac{g \sin\theta}{2\mu_{tin}} \left[-\rho_{tin} z^2 + 2(\rho_{glass} z_2 + \rho_{tin} z_1) z\right].$$

With all of the constants, properties, layer thicknesses, etc. inserted, this becomes:

$$u_{glass} = -547.22 (\mathrm{ms})^{-1} z'^2 + 0.76438 \frac{\mathrm{m}}{\mathrm{s}},$$

$$u_{tin} = -399017 (\mathrm{ms})^{-1} z^2 + 1162.85 \mathrm{s}^{-1} z$$

and when graphed (which wasn't required), this looks like:



(d) The maximum velocities are easy, just plug the top z for each layer into the velocity equations above:

$$u_{max,glass} = 0.76438,\ u_{max,tin} = 0.76383.$$

The average velocities are a bit less straightforward, we have to integrate the velocity over the layer:

$$u_{av} = \frac{Q}{A} = \frac{\int_{z_a}^{z_b} u_x W dz}{W(z_b - z_a)} = \frac{1}{z_b - z_a}\int_{z_a}^{z_b}\left(-\frac{g\sin\theta}{2\nu}z^2 + Az + B\right)dz,$$

$$u_{av} = \frac{1}{z_b - z_a}\left[-\frac{g\sin\theta}{6\nu}z^3 + \frac{A}{2}z^2 + Bz\right]_{z_a}^{z_b}.$$

Inserting the constants for each material gives:

$$u_{av,glass} = \frac{1}{0.001\text{m}}\left[-182.41(\text{ms})^{-1}z'^3 + 0.76438\frac{\text{m}}{\text{s}}z'\right]_{0.001\text{m}}^{0.002\text{m}} = 0.76420\frac{\text{m}}{\text{s}},$$

$$u_{av,tin} = \frac{1}{0.001\text{m}}\left[-133006(\text{ms})^{-1}z^3 + 581.42\text{s}^{-1}z^2\right]_{0}^{0.001\text{m}} = 0.4484\frac{\text{m}}{\text{s}}.$$

(e) In the glass, using the thickness of the glass sheet for L, the Reynolds number is about 2.4, which is quite stable. In the tin, it is about 1050, which would make the tin unstable if on its own. But since it is confined by the much more viscous glass, it acts like Couette flow, with a higher critical Reynolds number, so it can (barely) exist as a stable laminar flow.

5. Settling of magnesium hydroxide particles in water

(a) For a sphere of diameter d in a fluid, the net gravity/buoyancy force is given by:

$$F_w - F_b = \frac{1}{6}\pi d^3 g(\rho_s - \rho)$$

where ρ_s is the density of the sphere, and ρ that of the fluid. When the sphere is rising or sinking at its terminal velocity V, that force is exactly balanced by the drag force, given by:

$$|F_w - F_b| = F_d = fKA = f \cdot \frac{1}{2}\rho V^2 \cdot \frac{1}{4}\pi d^2$$

where the friction factor f in Stokes flow is:

$$f = \frac{24}{\text{Re}} = \frac{24\mu}{\rho V d}$$

resulting in a drag force of:

$$F_d = 3\pi\mu V d$$

We set this equal to $|F_w - F_b|$ and solve for V:

$$\frac{1}{6}\pi d^3 g|\rho_s - \rho| = |F_w - F_b| = F_d = 3\pi\mu V d$$

$$V = \frac{d^2 g|\rho_s - \rho|}{18\mu}$$

(b) Set $V = 10\text{cm}/1\text{min} = \frac{1}{600}\frac{\text{m}}{\text{s}}$ and solve for d:

$$d = \sqrt{\frac{18\mu V}{g|\rho_s - \rho|}} = \sqrt{\frac{18 \cdot 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}} \cdot \frac{1}{600}\frac{\text{m}}{\text{s}}}{9.8\frac{\text{m}}{\text{s}^2}|3850 - 1000|\frac{\text{kg}}{\text{m}^3}}}$$

$$d \simeq 47\mu\text{m}$$

All of the particles larger than 47μm across will sink faster than that, and reach the bottom within one minute.

(c) Since $V \propto d^2$, with half the diameter, the particles will sink at one quarter of the velocity. If they are initially uniformly dispersed throughout the suspension, only one quarter of them will reach the bottom within that time.

(d) The Reynolds number is:

$$\text{Re} = \frac{\rho V d}{\mu} = \frac{1000\frac{\text{kg}}{\text{m}^3} \cdot \frac{1}{600}\frac{\text{m}}{\text{s}} \cdot 3.3 \times 10^{-5}\text{m}}{10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}} = 0.079$$

This is below 0.1, so we're safely in the Stokes law régime. The smaller particle has half the diameter and one quarter the velocity, so one eighth the Reynolds number, which will also satisfy the Stokes law criterion.