18.014 Homework 9 - Solutions

Problem 1. P. 399: 15.
We will use the integral test. We will integrate by substitution, letting $u = \log(\log x)$. Then $du = \frac{1}{x \log x} dx$.

$$\int_3^n \frac{dx}{x \log x (\log(\log x))^s} = \int_{u_3}^{u_n} \frac{du}{u^s} = \begin{cases} \dfrac{(\log(\log n))^{-s+1} - (\log(\log 3))^{-s+1}}{-s+1} & \text{if} \quad s \neq 1 \\ \log(\log(\log n)) - \log(\log(\log 3)) & \text{if} \quad s = 1 \end{cases}$$

The value of this integral is finite when $s > 1$ and infinite when $s \leq 1$. Hence, the series is convergent when $s \geq 1$ and divergent when $s < 1$.

Problem 2.
a) P. 402: 3.
We will use the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} 2\Big(\frac{n}{n+1}\Big)^n = 2 \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{2}{e} < 1$$

Hence, the series is convergent.

b) P. 402: 12.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n} = \exp\left(\lim_{n\to\infty} \Big(n + \frac{1}{n}\Big) \log n - n \log\Big(n + \frac{1}{n}\Big)\right)$$
$$= \exp\left(\lim_{n\to\infty} \left(n\Big(\log n - \log\Big(n + \frac{1}{n}\Big)\Big) + \frac{\log n}{n}\right)\right)$$

Now,
$$\lim_{n\to\infty} \frac{\log n}{n} = 0$$
and
$$\lim_{n\to\infty} n\Big(\log n - \log\Big(n + \frac{1}{n}\Big)\Big) = \lim_{n\to\infty} \frac{\log\left(\frac{1}{1+\frac{1}{n^2}}\right)}{\frac{1}{n}} = 0$$

Hence,

$$\lim_{n\to\infty} a_n = \exp(0) = 1$$

The limit of $\{a_n\}$ is not 0, which is a necessary condition for a series to converge. Hence, the series is divergent.

Problem3.

a) P. 409: 11.

The example of page 398 of the text show that $\Sigma \frac{1}{n(\log n)^2}$ converges. For every $n \geq 2$, $0 < \frac{1}{n(\log n)^2} < \frac{1}{n(\log(n+1))^2}$. Then by the comparison test, $\Sigma \frac{1}{n(\log(n+1))^2}$ converges. Hence $\Sigma \frac{(-1)^n}{n(\log(n+1))^2}$ converges absolutely.

b) P. 410: 22.

$$\sin\left(n\pi + \frac{1}{\log n}\right) = \sin n\pi \cos \frac{1}{\log n} + \cos n\pi \sin \frac{1}{\log n} = (-1)^n \sin \frac{1}{\log n}$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sin \frac{1}{\log n} = \sin \lim_{n\to\infty} \frac{1}{\log n} = \sin 0 = 0$$

By Leibniz's rule, the alternating series converges.
Now, compare $\Sigma \sin \frac{1}{\log n}$ with $\Sigma \frac{1}{\log n}$.

$$\lim_{n\to\infty} \frac{\sin \frac{1}{\log n}}{\frac{1}{\log n}} = 1$$

$\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 2$. Since $\Sigma \frac{1}{n}$ diverges, then, $\Sigma \frac{1}{\log n}$ diverges and then $\Sigma \sin \frac{1}{\log n}$ diverges as well.
Then, the series is conditionally convergent.