18.014 Homework 10 - Solutions

Problem 1.
a)
Let a_n denote the length of the n-th segment. Let d be the spacing between two consecutive segments. We want to know whether Σa_n converges or not. First, we will define a_n:

$$a_n = \frac{1}{(1+nd)^2} - \frac{1}{(1+nd)^3} = \frac{nd}{(1+nd)^3}$$

Now, we will compare a_n with $\frac{1}{(nd)^2}$.

$$\lim_{n\to\infty} \frac{\frac{nd}{(1+nd)^3}}{\frac{1}{(nd)^2}} = \lim_{n\to\infty} \frac{(nd)^3}{(1+nd)^3} = \lim_{n\to\infty} \Big(\frac{1}{1+\frac{1}{nd}}\Big)^3 = 1$$

Since $\frac{1}{d^2}\Sigma\frac{1}{n^2}$ converges, then Σa_n converges. Hence the sum is finite.
b)

$$a_n = \frac{1}{(1+nd)} - \frac{1}{(1+nd)^3} = \frac{(nd)^2+2nd}{(1+nd)^3}$$

Now, we will compare a_n with $\frac{1}{nd}$.

$$\lim_{n\to\infty} \frac{\frac{(nd)^2+2nd}{(1+nd)^3}}{\frac{1}{nd}} = \lim_{n\to\infty} \frac{(nd)^3+2(nd)^2}{(1+nd)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{nd}}{\Big(\frac{1}{nd}+1\Big)^3} = 1$$

Since $\frac{1}{d}\Sigma\frac{1}{n}$ diverges, then Σa_n diverges too . The sum is infinite.

Problem 2.
a) P. 415: 8.
Compare to $\Sigma n^{s+\frac{1}{2}}$. You will get that the series converges for $s < \frac{1}{2}$ and diverges otherwise.
b) P. 415: 9.
Compare to Σn^{-2}. You will get that the series converges.

Problem 3. Solve the integral by partial fractions. You will get:

$$\int_1^y \Big(\frac{2x^2+bx+a}{x(2x+a)} - 1\Big)dx = \ln y + \frac{b-a-2}{2}(\ln(2y-a) - \ln(2+a))$$

Note that if $a > b$ the limit as $y \to \infty$ will go to 0, and if $b > a$ the limit will go to infinity. So $a = b$ and solving gives $a = b = 2e - 2$.

Problem 4. Using the root test, you get that the radius of convergence is $\frac{1}{e}$. Simple inspection shows that for $z = \pm\frac{1}{e}$ it diverges, since the limit is not 0.

Problem 5. Ratio test shows that the radius of convergence is $\frac{1}{2}$. Inspection shows that for $x = \pm\frac{1}{2}$ it diverges. Using the integral, you get that the series converges to $\frac{1}{1+2x} + \frac{\ln(1+2x)}{2x}$.