18.014 Homework 2 - Solutions

Problem 1. P.28: 6.

$$x < y$$
$$0 < y - x$$

By theorem I30, $\exists n \in \mathbb{Z}_+$ such that:

$$1 < n(y - x)$$
$$1 < ny - nx$$
$$nx + 1 < ny$$

According to exercise 5, page 28 (solved in recitation)$\exists m \in \mathbb{Z}$ such that:

$$nx < m \leq nx + 1 < ny$$
$$nx < m < ny$$
$$x < \frac{m}{n} < y$$

Since m, $n \in \mathbb{Z}$, $\frac{m}{n} \in \mathbb{Q}$.

Following the same argument, there must exist a rational number between x and $\frac{m}{n}$. This process can be carried on indefinitely, hence there must be an infinite number of rational numbers between x and y.

Problem 2. P.64: 7.

(a) From exercise 6, we know that the number of lattice points in S=$\{(x, y)$ s.t. $0 < x < b$, $0 < y \leq x \cdot \frac{a}{b}\}$ is equal to $\sum_{n=1}^{b-1}[\frac{na}{b}]$. S is the set of all the points inside the right triangle of sides a and b. Note that no lattice points lie on the hypotenuse of the triangle, since $\frac{na}{b}$ is an integer if and only if n is a multiple of b (because a and b have no common factor), and $1 \leq n \leq b-1$. Hence, the number of lattice points is equal to half the number of lattice points in the rectangle of sides a and b (since there are no lattice points on the diagonal), and that is $\frac{(a-1)(b-1)}{2}$.

(b) Since a and b have no common factors, $\frac{na}{b}$ is not an integer for $1 \leq n \leq b - 1$. Hence, by exercise 4:

$$[\frac{-na}{b}] = -[\frac{na}{b}] - 1$$

Therefore:

$$\begin{aligned}
\sum_{n=1}^{b-1}[\frac{na}{b}] &= \sum_{n=1}^{b-1}[\frac{a(b-n)}{b}] \\
&= \sum_{n=1}^{b-1}[a-\frac{na}{b}] \\
&= \sum_{n=1}^{b-1}\Big(a+[-\frac{na}{b}]\Big) \\
&= \sum_{n=1}^{b-1}\Big(a-[\frac{na}{b}]-1\Big) \\
&= (a-1)(b-1)-\sum_{n=1}^{b-1}[\frac{na}{b}] \\
\Rightarrow 2\cdot\sum_{n=1}^{b-1}[\frac{na}{b}] &= (a-1)(b-1) \\
\Rightarrow \sum_{n=1}^{b-1}[\frac{na}{b}] &= \frac{(a-1)(b-1)}{2}
\end{aligned}$$

Problem 3.

Let $s(x)$ be a step function defined on the partition $P = \{x_0, x_1, ..., x_n\}$, such that $s(x) = s_k \forall x_{k-1} < x < x_k$. I will add a new subdivision point, y, such that $x_{k-1} < y < x_k$. Therefore, $s(y) = s_k$. With the new partition the term $s_k^3(x_k - x_{k-1})$ is replaced by:

$$\begin{aligned}
s_k^3(x_k-y)+s_k^3(y-x_{k-1}) &= s_k^3x_k - s_k^3y + s_k^3y - s_k^3x_{k-1} \\
&= s_k^3x_k - s_k^3x_{k-1} \\
&= s_k^3(x_k - x_{k-1})
\end{aligned}$$

Hence, the addition of any subdivision point doesn't alter the summation, so it doesn't alter the integral, proving that the integral is independent of the partition.

(a)

$$\int_a^b s(x)dx = \sum_{k=1}^{m} s_k^3(x_k - x_{k-1}) \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_m = b$$

$$\int_b^c s(x)dx = \sum_{k=1}^{n-m} s_k^3(y_k - y_{k-1}) \quad \text{with} \quad b = y_0 < y_1 < \cdots < y_{n-m} = c$$

Let $y_i = x_{m+i}$. Then $P = \{x_0, x_1, ..., x_n\}$ is a partition for $[a, c]$, in which $s(x)$ is a step function. Then:

$$\begin{aligned}\int_a^b s(x)dx + \int_b^c s(x)dx &= \sum_{k=1}^{m} s_k^3(x_k - x_{k-1}) + \sum_{k=m+1}^{n} s_k^3(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} s_k^3(x_k - x_{k-1}) \\ &= \int_a^c s(x)dx\end{aligned}$$

(b)

Let $s(x) = 1$ $t(x) = 2$ $\forall x \in [0, 1]$. Then:

$$\int_a^b (s + t) = (1 + 2)^3(1 - 0) = 27 \neq 9 = 1^3(1 - 0) + 2^3(1 - 0) = \int_a^b s + \int_a^b t$$

(c)

Let $s(x) = 1$, $c = 2$ $\forall x \in [0, 1]$. Then:

$$\int_a^b cs = (1 \cdot 2)^3(1 - 0) = 8 \neq 2 = 2 \cdot (1)^3(1 - 0) = c \cdot \int_a^b s$$