18.014 Homework 3 - Solutions

Problem 1. P.83: 22 (b).

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^c f(x)dx + \int_c^1 f(x)dx \\
&= \int_0^c xdx + \int_c^1 c\frac{1-x}{1-c}dx \\
&= \frac{c^2}{2} + \frac{c}{1-c} \cdot \int_c^1 dx - \frac{c}{1-c} \cdot \int_c^1 xdx \\
&= \frac{c^2}{2} + c - \frac{c}{1-c} \cdot \frac{1^2}{2} + \frac{c}{1-c} \cdot \frac{c^2}{2} \\
&= \frac{c}{2}
\end{aligned}$$

Problem 2. P.94: 16.

First, we need to know for which x it is true that $f(x) \geq g(x)$.
The curves intersect when:

$$\begin{aligned} f(x) &= g(x) \\ x - x^2 &= ax \\ x(x + a - 1) &= 0 \end{aligned}$$

Then, $x = 0$ or $x = 1 - a$.
Hence $f(x) \geq g(x)$ whenever x is between 0 and $1 - a$. (Note that $1 - a$ can be positive or negative).

$$\begin{aligned} \int_0^{1-a} (f(x) - g(x))dx &= \pm\frac{9}{2} \\ \int_0^{1-a} ((1-a)x - x^2)dx &= \pm\frac{9}{2} \\ \frac{(1-a)^3}{2} - \frac{(1-a)^3}{3} &= \pm\frac{9}{2} \\ \frac{(1-a)^3}{6} &= \pm\frac{9}{2} \\ (1-a)^3 &= \pm 27 \\ 1 - a &= \pm 3 \\ a &= -2 \\ a &= 4 \end{aligned}$$

Problem 3. P. 114: 10.

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\begin{aligned}
V &= \int_0^{\frac{\pi}{4}} \pi(f(x)^2 - g(x)^2)dx \\
&= \int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)dx \\
&= \pi \int_0^{\frac{\pi}{4}} \cos(2x)dx \\
&= \frac{\pi}{2}[\sin(2x)]|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2}
\end{aligned}
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