<u>Exercises</u> on derivatives

1. Define a new derivative by the formula

$$D^{\#}f(x) = \lim_{h\to 0} \frac{(f(x+h))^3 - (f(x))^3}{h} .$$

   Assuming that f and g are continuous, and that $D^{\#}f(x)$ and $D^{\#}g(x)$ exist, derive formulas for $D^{\#}(f(x)g(x))$ and $D^{\#}(1/f(x))$ in terms of $D^{\#}f(x)$ and $D^{\#}g(x)$.

2. Define a new derivative by the formula

$$D^{*}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h^2} .$$

   If $f(x) = x^2 + 3$, show that $D^{*}f(x)$ exists only at the point $x = 0$, and compute $D^{*}f(0)$.

3. Assume the usual properties of the sine and cosine functions. Define

$$f(x) = \begin{cases} x \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

   (a) Apply the definition of derivative to determine whether $f'(0)$ and $g'(0)$ exist. Compute them if they do exist.

   (b) Show that $f'(x)$ and $g'(x)$ are not continuous at $x = 0$. Explain which part of the definition of continuity is violated in each case.

4. If $f(x) = u(v(x))$, write down a formula for $f''(x)$, assuming u', u'', v', and v'' exist at the points in question.

5. Suppose $f(x)$ is continuous and strictly monotonic on the interval $[a,b]$; let $g(y)$ be its inverse function. Show that if f' and f'' exist on $[a,b]$, then g'' exists at each point y for which $f'(g(y)) \neq 0$, and

$$g''(y) = -\frac{f''(g(y))}{[f'(g(y))]^3}.$$

6. Let $f(x) = 2x^5 - 5x^4 + 5$ for $x \geq 2$; let $g(y)$ be the inverse function to f. Let c be the number for which $f(c) = 0$. (See Exercise 3 of Section G.)

(a) Note that $g(0) = c$; show that $g(-11) = 2$ and $g(86) = 3$.

(b) Show that

$$g'(0) = \frac{1}{10c^3(c-2)}.$$

(c) Compute $g'(-11)$ and $g'(86)$.

7. Suppose f is a function defined for all x such that:

$f(1) = 2$ and $f(2) = 3$ and $f(3) = 4$;

$f'(1) = 6$ and $f'(2) = 10$ and $f'(3) = 7$;

$f''(1) = 3$ and $f''(2) = 2$ and $f''(3) = 1$.

(a) Let $h(x) = f(f(x))$; compute $h(1)$, $h'(1)$, and $h''(1)$. (Answers: 3, 60, 102.)

(b) Suppose f is strictly increasing. Let $g(y)$ be its inverse function, and compute $g(3)$, $g'(3)$, and $g''(3)$. (Answers: 2, 1/10, -1/500.)

8. Derive a formula for the derivative of $\sqrt{x}$ directly from the definition.

9. Using the fact that $f(x) = \sqrt[3]{x}$ is defined and continuous for all x, derive a formula for $f'(x)$, when $x \neq 0$, directly from the definition of the derivative.

[Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. Let $a = \sqrt[3]{x+h}$ and $b = \sqrt[3]{x}$.]