18.014 Homework 8 - Solutions

Problem 1.
Using integration by parts with $u = x$ and $v = \frac{1}{(x^2+a^2)^{(n-1)}}$:

$$\begin{aligned} I_{n-1} = \int \frac{dx}{(x^2+a^2)^{n-1}} &= \frac{x}{(x^2+a^2)^{(n-1)}} + \int \frac{2(n-1)x^2dx}{(x^2+a^2)^n} \\ &= \frac{x}{(x^2+a^2)^{(n-1)}} + 2(n-1)\int \frac{(x^2+a^2)dx}{(x^2+a^2)^n} - 2(n-1)\int \frac{a^2dx}{(x^2+a^2)^n} \\ &= \frac{x}{(x^2+a^2)^{(n-1)}} + 2(n-1)I_{n-1} - 2(n-1)a^2 I_n \end{aligned}$$

$$\Rightarrow I_n = \frac{1}{2a^2(n-1)}\left[\frac{x}{(x^2+a^2)^{(n-1)}} + (2n-3)I_{n-1}\right]$$

Problem 2.
Using Taylor's theorem:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6) \\ e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + o(x^5) \\ (\sin x)^2 &= x^2 + o(x^3) \\ \sin x^3 &= x^3 + o(x^4) \end{aligned}$$

Hence:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x - xe^{x^2} + \frac{7x^3}{6}}{(\sin x)^2 \sin x^3} &= \lim_{x\to 0} \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - x - x^3 - \frac{x^5}{2} + \frac{7x^3}{6} + o(x^6)}{(x^2+o(x^3))(x^3+o(x^4))} \\ &= \lim_{x\to 0} \frac{-\frac{59x^5}{120} + o(x^6)}{x^5 + o(x^6)} \\ &= \lim_{x\to 0} \frac{-\frac{59}{120} + o(x)}{1 + o(x)} \\ &= -\frac{59}{120} \end{aligned}$$

Problem 3.
Using Taylor's theorem:

$$\begin{aligned}
e^{\frac{x}{a}} &= e^0 + e^0\frac{x}{a} + \frac{e^0}{2}\left(\frac{x}{a}\right)^2 + \cdots \\
&= 1 + \frac{x}{a} + \frac{1}{2}\left(\frac{x}{a}\right)^2 + o\left(\left(\frac{x}{a}\right)^3\right) \\
\sqrt{\frac{a+x}{a-x}} &= \frac{a+0}{a-0} + \left(\frac{a-0}{a+0}\right)^{\frac{1}{2}}\frac{a}{(a-0)^2}x + \frac{1}{2}\frac{2a(0)+a^2}{(a-0)^4}\left(\frac{a-0}{a+0}\right)^{\frac{3}{2}}x^2 + \cdots \\
&= 1 + \frac{x}{a} + +\frac{1}{2}\left(\frac{x}{a}\right)^2 + o\left(\left(\frac{x}{a}\right)^3\right)
\end{aligned}$$

So they agree up to terms of order $\frac{x^2}{a^2}$.