18.014 Homework 5 - Solutions

Problem 1.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \cdot \frac{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h[(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}}]} \\
&= \lim_{h\to 0} \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}}} \\
&= \frac{1}{3x^{\frac{2}{3}}}
\end{aligned}
$$

Problem 2.

$$
\begin{aligned}
f(x) &= \frac{\tan^6 x + 9\tan^4 x - 9\tan^2 x - 1}{3\tan^3 x} \\
f'(x) &= \frac{3\tan^3 x(6\tan^5 x + 36\tan^3 x - 18\tan x) - (9\tan^2 x)(\tan^6 x + 9\tan^4 x - 9\tan^2 x - 1)}{9\tan^6 x \cdot \cos^2 x} \\
&= \frac{\tan^6 x + 3\tan^4 x + 3\tan^2 x + 1}{\tan^4 x \cos^2 x} = \frac{(\tan^2 x + 1)^4}{\tan^4 x \cos^2 x} = \frac{1}{\sin^4 x \cos^4 x}
\end{aligned}
$$

Problem 3.

◇ **Critical points:** There are no points where $f'(x) = 0$. $f'(x)$ doesn't exist when $x = 0$.

◇ **Zeros:** $x = \pm 3^{\frac{1}{4}}$.

◇ **Asymptotes:** $x = 0$.

◇ **Intervals of monotonicity:** It is increasing in the separate intervals $(-\infty, 0)$ and $(0, \infty)$. (Note that this is different than $(-\infty, 0) \bigcup (0, \infty)$).

◇ **Convexity:** Convex in $[-1, 0)$ and $[1, \infty]$. Concave in $(-\infty, -1]$ and $(0, 1]$.

◇ **Points of inflection:** $f''(x) = 0$ at $x = \pm 1$. $f''(x)$ doesn't exist at $x = 0$.