

18.014 Homework 1 - Solutions

Problem 1

Theorem I.6.

$$\begin{aligned} a \cdot 0 &= 0 \cdot a && \text{Commutativity} \\ 0 &= b - b && \text{Definition of } -b \\ a \cdot 0 &= a \cdot (b - b) \\ &= a \cdot b - a \cdot b && \text{Theorem I.5} \\ &= 0 && \text{Theorem I.2} \end{aligned}$$

Then $a \cdot 0 = 0 \cdot a = 0$

Theorem I.11.

For the sake of argument, assume $a \neq 0$.
Hence, there exists a^{-1} , such that $a^{-1} \cdot a = 1$.

$$\begin{aligned} ab &= 0 \\ a^{-1} \cdot (ab) &= a^{-1} \cdot 0 \\ (a^{-1} \cdot a) \cdot b &= 0 && \text{Associativity and theorem I.6.} \\ 1 \cdot b &= 0 \\ b &= 0 && \text{Identity for multiplication} \end{aligned}$$

The same proof can be reproduced for b . Hence, if $ab = 0$, $a = 0$ or $b = 0$.

Problem 2

P.36: 5.

We will prove by induction that $\prod_{k=2}^n (1 - \frac{1}{k^2}) = \frac{n+1}{2n}$

When $n = 2$:

$$1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$$

Now, we will assume it is true for a fixed n , and show that then is true for $n + 1$. Suppose

$$\begin{aligned}\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) &= \frac{n+1}{2n} \\ \prod_{k=2}^{n+1} \left(1 - \frac{1}{k^2}\right) &= \left(\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right)\right) \left(1 - \frac{1}{(n+1)^2}\right) \\ &= \left(\frac{n+1}{2n}\right) \left(1 - \frac{1}{(n+1)^2}\right) \\ &= \frac{n+2}{2(n+1)}\end{aligned}$$

I skipped some steps between the last two inequalities, because I know you are able to do them. The proof is complete by induction.

P.36: 6

(a) Suppose $\sum_{i=1}^k i = \frac{1}{8}(2k+1)^2$

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) \\ &= \frac{1}{8}(2k+1)^2 + (k+1) \\ &= \frac{4k^2 + 4k + 1 + 8k + 8}{8} \\ &= \frac{4k^2 + 12k + 9}{8} \\ &= \frac{1}{8}(2(k+1) + 1)^2\end{aligned}$$

Therefore, if $A(k)$ is true, $A(k+1)$ must be true.

(b) Let $n = 1$.

$$1 \neq \frac{9}{8} = \frac{(2 \cdot 1 + 1)^2}{8}$$

The proof doesn't work because the statement is not true for the base.

(c) The equality can be replaced by:

$$\sum_{i=1}^k i < \frac{1}{8}(2k+1)^2$$

This can be proved by induction.

For $n = 1$ we have $1 < \frac{9}{8}$.

The step of the induction is similar to part (a), you should be able to do it.

Problem 3.

We will proceed by induction. For the proof we will use the following identities:

$$\begin{aligned} C_n^{k-1} + C_n^k &= C_{n+1}^k \\ C_n^0 &= C_n^n = 1 \end{aligned}$$

You can prove those statements by yourself, you just need a little bit of algebra.

When $n = 1$:

$$\begin{aligned} (a+b)^1 &= a+b = \frac{1!}{1!0!}a^1b^0 + \frac{1!}{0!1!}a^0b^1 \\ &= \sum_{k=0}^1 C_1^k a^k b^{1-k} \end{aligned}$$

Now, the step of the induction. Suppose

$$(a+b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k}$$

Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n \cdot (a+b) \\
&= \left(\sum_{k=0}^n C_n^k a^k b^{n-k} \right) \cdot (a+b) \\
&= \sum_{k=0}^n C_n^k a^{k+1} b^{n-k} + \sum_{k=0}^n C_n^k a^k b^{n-k+1} \\
&= \sum_{k=1}^{n+1} C_n^{k-1} a^k b^{n-k+1} + \sum_{k=0}^n C_n^k a^k b^{n-k+1} \\
&= C_n^n a^{n+1} + \sum_{k=1}^n C_n^{k-1} a^k b^{n-k+1} + \sum_{k=1}^n C_n^k a^k b^{n-k+1} + C_n^0 b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^n (C_n^{k-1} + C_n^k) a^k b^{n-k+1} + C_n^0 b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^n C_{n+1}^k a^k b^{n-k+1} + C_n^0 b^{n+1} \\
&= \sum_{k=0}^{n+1} C_{n+1}^k a^k b^{n-k+1}
\end{aligned}$$

The proof is completed by induction.