18.014 Homework 4 - Solutions

Problem 1. P.155: 6.
Since $f(x)$ is continuous, we can apply the mean-value theorem for integrals. We can assume that $a \neq b$, if not, the interval $[a, b]$ would be trivial. Hence, for some $c \in [a, b]$ we have:

$$\int_a^b f(x)dx = f(c) \cdot (b - a)$$
$$0 = f(c) \cdot (b - a)$$
$$0 = f(c)$$

Problem 2. P.149: 5.
If $x < y < 1$, $f(x) = x < y = f(y) < 1$.
If $1 \leq x < y \leq 4$, $f(x) = x^2 < xy < y^2 = f(y)$, thus $1 \leq f(x) < f(y) \leq 16$.
If $4 < x < y$, $8 \cdot 4^{\frac{1}{2}} < f(x) = 8x^{\frac{1}{2}} < 8y^{\frac{1}{2}} = f(y)$.
Note that $x^{\frac{1}{2}}$ is increasing because it is the inverse function of x^2, which is continuous and strictly increasing.
Then, if $x < y$, $f(x) < f(y)$. Hence, $f(x)$ is strictly increasing. The domain of g is the range of f, that is $\mathbf{R}$.

$$g(y) = \begin{cases} y & \text{if} \quad y < 1 \\ y^{\frac{1}{2}} & \text{if} \quad 1 \leq y \leq 16 \\ \dfrac{y^2}{64} & \text{if} \quad y > 16 \end{cases}$$

Problem 3.
(a) Consider the interval $[0, b]$. By the intermediate value theorem, f takes every value between $f(0)$ and $f(b)$. This is true for every b nonnegative, hence f takes every value between $f(0)$ and M.
Now, suppose that $M = f(c)$ for some c. Take $d > c$. Since f is strictly increasing, $f(d) > f(c) = M$, which contradicts the hypothesis of M being the supremum.
(b) f takes any value between $f(0)$ and M. We shall consider $\varepsilon < M - f(0)$ (The other case is obvious). Then, given ε, $\exists c$ such that $f(c) + \varepsilon = M$.
Since f is continuous, by the small-span theorem, it is uniformly continuous

in $[0, c]$. Hence, for ε $\exists\delta$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.
If $c < x$, $\forall a > c$, $|f(x) - f(a)| < M - f(c) = \varepsilon$.
Especially, $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.
f is uniformly continuous.