

18.01 Problem Set 5

Due Friday 10/27/06, 1:55 pm

Part I (20 points)

Lecture 19. Fri. Oct. 20 First fundamental theorem. Properties of integrals.

Read: 6.6, 6.7 (The second fundamental theorem, discussed in Lecture 20, is stated in the text as (13) at the bottom of page 215 and also as Step 1, page 207, of the proof of the first fundamental theorem.)

Work: 3C-1, 2a, 3a, 5a; 3E-6bc; 4J-2

Lecture 20. Tues. Oct. 24 Second Fundamental Theorem. Def'n of $\ln x$.

Read: Notes PI, p.2 [eqn.(7) and example]; Notes FT.

Work: 3E-1, 3a; 3D-1, 4bc, 5, 8a; 3E-2ac

Lecture 21. Thur. Oct. 26 Areas between curves. Volumes by slicing.

Read: 7.1, 7.2, 7.3 Work¹: 4A-1b, 2, 4; 4B-1de, 6, 7

Lecture 22. Fri. Oct. 27 Volumes by disks and shells.

Read: 7.4 Work on PS 6

Part II (30 points + 5 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”. (See full explanation on PS1).

1. (Lec 19, 6 pts: 3 + 3) (+ 5 extra for part (c))

a) Suppose that at the beginning of day 0, some time last summer, the temperature in Boston was $y(0) = 65^\circ$ Fahrenheit and that over a 50-day period, the temperature increased according to the rule $y'(t) = y(t)/100$, with time t measured in days. Find the formula for y , and draw a graph of temperature on days 3 and 4, $3 \leq t \leq 5$, and label with the correct day and shade in the regions whose areas represent the average temperature each of the two days.²

¹A more colorful way of expressing 4B-6 is in terms of the volume of a tent, as in the textbook problem 7.3/7. Unfortunately, the problem is ill-posed and can't be done without pretending that the cross-sections are triangles. In real life, the canvas would have creases. In general, the shapes formed by stretching canvas or nylon over various arrays of tent poles are quite hard to compute.

²The continuous average of a function is

$$\frac{1}{b-a} \int_a^b f(x) dx$$

In this case $b - a = 1$, so the average is the same as integral. For more, see Notes, AV, and Lecture 23.

b) The number of *cooling degree days* is the sum for each day of the difference between the average temperature for that day and 65° . The number is used to estimate the demand for electricity for air conditioning. Draw a second graph of y for the whole 50 days and shade in the region whose area represents the total number of degree days. Write a formula for this total area as the difference between $65 \cdot 50$ and a definite integral. Evaluate the definite integral using the fundamental theorem of calculus. (Alternatively, write the whole quantity as an integral expressing the area between curves as in Lecture 21, and 7.2.)

c) (extra credit) Compute the definite integral in part (b) directly by evaluating a lower Riemann sum and taking a limit. Follow the procedure in Problem 5, PS4, but with different scale factors. This rather elaborate calculation shows how much time and effort we save every time we use the fundamental theorem and the change of variable formula in integrals.

2. (Lec 20, 16 pts: $2 + 2 + 4 + 2 + 6$) Consider the function $f(x) = \int_0^x \cos(t^2)dt$. There is no expression for $f(x)$ in terms of standard elementary functions. It is known as a Fresnel integral, along with the corresponding sine integral.

a) Draw a rough sketch of $\cos(t^2)$, showing the first positive and negative zeros. What does the curve look like at $t = 0$? Is the function even or odd?

b) List the critical points of $f(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

c) Sketch the graph of f on the interval $-2 \leq x \leq 2$, with labels for the critical points **and inflection points**. (The drawing should be qualitatively correct, but just estimate the values of f at the labelled points.)

d) Estimate $f(0.1)$ to six decimal places.

e) Fresnel integrals are sometimes expressed using different scaling of the variables.

i) Let $g(x) = \int_0^x \cos((\pi/2)u^2)du$. Make a change of variables to show that $f(x) = c_1g(c_2x)$ for some constants c_1 and c_2 . Why did we choose the factor $\pi/2$?

ii) Let $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}}dv$. (This integral is called *improper* because $1/\sqrt{v}$ is infinite³ at $v = 0$.) Make a different change of variable to show that $f(x) = ch(x^2)$ for some constant c (assume that $x > 0$).

iii) Let $k(x) = \sqrt{x} \int_0^1 \cos(xt^2)dt$, $x > 0$. Use the change of variable $z = xt^2$ and part (ii) to find the relationship between the functions k and f . Hint: Which quantities are variable and which are constant? (Not assigned: you can also work out this relationship using a change of variables like the one in part (i).)

3. (Lec 21, 5 pts: $2 + 3$)

a) Do 7.3/22.

b) Find the volume of the region in 3-space with $x > 0$, $y > 0$ and $z > 0$ given by

$$z^2/2 < x + y < z$$

Hint: First find the area of the horizontal cross-sections.

³Although the integrand is infinite, the area under the curve is finite. The function h is continuous, $h(0) = 0$, but its graph has infinite slope at $x = 0$.