Problem Set IV Solutions

1. $\oint 9.2$ 7.

   Let $f(x,y) = z = e^{3xy}$ and let P be the point of coordinates $(1,-2,e^{-6})$. The vector $\vec{N}_P = -\frac{\partial f}{\partial x}|_P \hat{\imath} - \frac{\partial f}{\partial y}|_P \hat{\jmath} + \hat{k}$ is normal to the graph of the surface. Differentiating we get $\vec{N}_P = 6e^{-6}\hat{\imath} - 3e^{-6}\hat{\jmath} + \hat{k}$. Hence the equation of the plane tangent to the surface is $6e^{-6}x - 3e^{-6}y + z = a$ for some $a \in \mathbb{R}$. Since P is in this plane, $a = 13e^{-6}$. Hence the equation of the plane is $6e^{-6}x - 3e^{-6}y + z = 13e^{-6}$.

2. $\oint 9.2$ 8.

   Let x, y, and z be the dimensions of the box, so that $x = 3 \pm 0.01, y = 4 \pm 0.01$, and $z = 12 \pm 0.03$. Let P be the point of coordinates (3,4,12). The length of the interior diagonal is given by the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. The partial derivates of f are: $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$, and $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$. Hence the length of the interior diagonal is $f(3,4,12) = 13$ cm, and the possible error is $\Delta f_{app} = \pm 0.01 \frac{\partial f}{\partial x}|_P \pm 0.01 \frac{\partial f}{\partial y}|_P \pm 0.03 \frac{\partial f}{\partial z}|_P = \pm \frac{43}{13} 0.01$.

3. $\oint 9.4$ 9.

   Let P be the point of coordinates (1,1,0) and let $\vec{G}_P = \frac{\partial f}{\partial x}|_P \hat{\imath} + \frac{\partial f}{\partial y}|_P \hat{\jmath} + \frac{\partial f}{\partial z}|_P \hat{k}$.From the given directional derivatives, it follows that $\vec{G}_P \cdot \hat{k} = 1$, $\vec{G}_P \cdot (\frac{-\hat{\jmath}-\hat{k}}{\sqrt{2}}) = \sqrt{2}$ and $\vec{G}_P \cdot (\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}) = 0$. Hence $\vec{G}_P \cdot (\frac{-\hat{\imath}-\hat{\jmath}}{\sqrt{2}}) = \frac{\sqrt{2}}{2}$, so $\frac{df}{ds} = \frac{\sqrt{2}}{2}$ at point $P = (1,1,0)$ in the direction towards $(0,0,0)$.

4. $\oint 9.6$ 6.

   The partial derivates of f are $\frac{\partial f}{\partial x} = y^3 - ze^{xz}, \frac{\partial f}{\partial y} = 3xy^2 + z^2$, and $\frac{\partial f}{\partial z} = -xe^{xz} + 2yz$. The unit tangent vector to $\vec{R}(t)$ is $\hat{T}(t) = \frac{\hat{\imath}+2t\hat{\jmath}+2t\hat{k}}{\sqrt{1+8t^2}}$. Hence $\hat{T}(1) = \frac{\hat{\imath}+2\hat{\jmath}+2\hat{k}}{3}$. So the directional derivative of f in the direction given by $\hat{T}(1)$ is $\frac{df}{ds} = (\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}) \cdot \hat{T}(1) = \frac{y^3}{3} + 2xy^2 + \frac{2}{3}z^2 + \frac{4}{3}yz - \frac{z+2x}{3}e^{xz}$.

5. $\oint$10.2 4.

   (a) $(\frac{\partial z}{\partial u})_{v,w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 2xvw + 2y(v+w)$.

   (b) If $u = 1, v = 2$, and $w = 3$, then $x = 6$ and $y = 11$.

   Hence $(\frac{\partial z}{\partial u})_{v,w} = 182$.

6. $\oint$10.2 6.

   (a) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = e^{2x+3y}(\frac{2}{t}\cos 4z + \frac{6t}{t^2+1}\cos 4z - 4\sin 4z)$.

   (b) If $t = \pi$, then $x = ln\pi$, $y = ln(\pi^2+1)$, $z = \pi$, and $w = \pi^2(\pi^2+1)^3$.

7. $\oint$10.2 9.

   Let $Q = (1,2)$ and $P = (g(1,2), h(1,2)) = (3,5)$.

   (a) We cannot find $f(3,5)$ with the given information.

   (b) For $x = 1$ and $y = 2$, $\frac{\partial w}{\partial x})_y = \frac{\partial w}{\partial u}|_P \frac{\partial u}{\partial x}|_Q + \frac{\partial w}{\partial v}|_P \frac{\partial v}{\partial x}|_Q = f_u(P)g_x(Q) + f_v(P)h_x(Q) = 27$.

   (c) For $x = 1$ and $y = 2$, $\frac{\partial w}{\partial y})_x = \frac{\partial w}{\partial u}|_P \frac{\partial u}{\partial y}|_Q + \frac{\partial w}{\partial v}|_P \frac{\partial v}{\partial y}|_Q = f_u(P)g_y(Q) + f_v(P)h_y(Q) = -31$.

8. $\oint$10.5 1.

   The rate at which the temperature is increasing is given by

$$\frac{\partial T}{\partial t} = T_x\frac{dx}{dt} + T_y\frac{dy}{dt} + T_z\frac{dz}{dt} + T\frac{dt}{dt} = T_x x'(t) + T_y y'(t) + T_z z'(t) + T_t.$$

9. $\oint$10.5 2.

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx} + \frac{dw}{dv}\frac{dv}{dx} = f_u h_x + f_v g_x$$

10. $\oint$10.5 6.

   Using the result from the previous problem, $\frac{dw}{dx} = f_u h_x + f_v g_x = 2vu - 2u$. If $x = 0$, then $u = 2$ and $v = 0$, so $\frac{dw}{dx} = -4$.

11. $\oint$ 10.5 10.

   (a) $(\frac{\partial w}{\partial x})_y = f_x\frac{dx}{dx} + f_y\frac{dy}{dx} + f_z\frac{dg}{dx} = f_x + f_z g_x$

(b) If $w = f(x, y, z) = xy+yz+zx$ and $z = g(x, y) = xy$, then $f_x = y+z$, $f_z = x+y$, and $g_x = y$. Hence $(\frac{dw}{dx})_y = y+z+(x+y)y = y+2xy+y^2$. Now let us check the result: $w = xy + yz + zx = xy + xy^2 + x^2y$, so $\frac{dw}{dx} = y + 2xy + y^2$.