Lecture XXX

n-Vectors and Matrices

1 n-Vectors

We define $\mathbf{E^n}$ to be the n-dimensional Euclidean space, and $\mathbf{R^n}$ to be the set of points in $\mathbf{E^n}$. Hence $\mathbf{R^n}$ is the set of all ordered n-tuples of real numbers $(x_1, \ldots, x_n)$. Ordered n-tuples allow repetitions, i.e. $x_i = x_j$ is possible for $i \neq j$, but if $x_i \neq x_j$, then $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \neq (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$. These n-tuples can be viewed as vectors in the n-dimensional Euclidean space. Hence we will call them *n-vectors.*

Definition 1 *Let* $P = (p_1, p_2, \ldots, p_n)$ *and* $Q(q_1, q_2, \ldots, q_n)$*. We say that the* distance between P and Q *is* $d(P,Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$.

We define $\vec{0} = (0, 0, \ldots, 0)$. On n-vectors, we can define operations analogous to the ones on 3-vectors:

1. Multiplication of a vector by a scalar

   Let $\vec{A} = (a_1, a_2, \ldots, a_n)$ and let k be a scalar. Then $k\vec{A} = (ka_1, ka_2, \ldots, ka_n)$.

2. Vector addition

   Let $\vec{A} = (a_1, a_2, \ldots, a_n)$ and $\vec{B} = (b_1, b_2, \ldots, b_n)$. Then $\vec{A} + \vec{B} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

3. Dot product

   Let $\vec{A} = (a_1, a_2, \ldots, a_n)$ and $\vec{B} = (b_1, b_2, \ldots, b_n)$. Then $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

4. Magnitude of a vector

   Let $\vec{A} = (a_1, a_2, \ldots, a_n)$. We say the magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

5. Unit vectors

Let $\vec{A} \neq \vec{0}$. Then $\hat{A} = \frac{1}{|\vec{A}|}\vec{A}$ has magnitude 1, hence it is called *unit vector.*

Definition 2 *We say that two nonzero vectors $\vec{A}$, $\vec{B}$ are* orthogonal *if $\vec{A} \cdot \vec{B} = 0$.*

For example, in $\mathbf{E^4}$, $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ play the role of $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ from $\mathbf{E^3}$. They are mutually orthogonal and any 4-vector as a linear combination of these vectors.

Definition 3 *Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$ be mutually orthogonal n-vectors. We say that they form a* frame.

Theorem 1 (Cauchy-Schwartz inequality) *For any given n-vectors $\vec{A}$ and $\vec{B}$, $|\hat{A} \cdot \hat{B}| \leq 1$.*

Theorem 2 (Triangle inequality) *For any given n-vectors $\vec{A}$ and $\vec{B}$, $|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$.*

Proof of Theorem 1:

Since $(\hat{A}-\hat{B})\cdot(\hat{A}-\hat{B}) \geq 0$, we get that $\hat{A}\cdot\hat{A} - 2\hat{A}\cdot\hat{B} + \hat{B}\cdot\hat{B} = 2 - 2\hat{A}\cdot\hat{B} \geq 0$, hence $\hat{A}\cdot\hat{B} \leq 1$. In the same manner, from $(\hat{A}-\hat{B})\cdot(\hat{A}+\hat{B}) \geq 0$, we get that $\hat{A}\cdot\hat{B} \geq -1$. Hence $|\hat{A}\cdot\hat{B}| \leq 1$.

Proof of Theorem 2:

From Theorem 1, we have that

$$\hat{A}\cdot\hat{B} \leq 1, \quad \textit{hence} \quad \frac{\vec{A}}{|\vec{A}|}\cdot\frac{\vec{B}}{|\vec{B}|} \leq 1, \quad \textit{so} \quad \vec{A}\cdot\vec{B} \leq |\vec{A}\cdot\vec{B}|.$$

Hence

$$|\vec{A}+\vec{B}|^2 = \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B} \leq |\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|^2 = (|\vec{A}| + |\vec{B}|)^2$$

So $|\vec{A}+\vec{B}| \leq |\vec{A}| + |\vec{B}|$.

Since $|\hat{A}\cdot\hat{B}| \leq 1$ and $\vec{A}$ and $\vec{B}$ are orthogonal when $\vec{A}\cdot\vec{B} = \hat{A}\cdot\hat{B} = 0$, we can define the *angle between $\vec{A}$ and $\vec{B}$* to be $\cos^{-1}(\hat{A}\cdot\hat{B})$.

2 Matrices

Definition 4 *An* $m \times n$ matrix *is a rectangular array of* m *rows and* n *columns of numbers.*

One-column matrices are called column-vectors and one-row matrices are called row-vectors. Generally, we write an $m \times n$ matrix A in the following way:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We can also write $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$. We define the following operations on matrices:

1. Multiplication with a scalar

   Let $A = (a_{ij})$ be an $m \times n$ matrix. Then $kA = (ka_{ij})$.

2. Matrix addition

   Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then $A + B = (a_{ij} + b_{ij})$.

Definition 5 *Let* A *be an* $m \times n$ *matrix. The* transpose *of* A *is an* $n \times m$ *matrix denoted by* A^T *such that its* i*-th column reading from top to bottom is the* i*-th row of* A *reading from left to right, for all* $i \leq m$*.*

For example,

$$if \quad A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}, \quad then \quad A^T = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 0 \end{bmatrix}.$$

The $m \times n$ matrix all whose entries are 0 is denoted by $0_{m,n}$. An $n \times n$ matrix is called a *square matrix.* An $n \times n$ matrix that has all elements on the main diagonal (from upper left to lower right) equal to 1 and all other elements equal to 0 is called an *identity matrix*, and is denoted by I_n. Let us define a new operation on matrices, *multiplication of a matrix with a column-vector*:

Let $A = (a_{i,j})$ be an $m \times n$ matrix, and let $B = (b_i)$ be an $n \times 1$ matrix. Then we define the $m \times 1$ matrix AB as follows:

$$AB = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \cdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix}$$