Lecture XXIX

Mathematical Applications

1 Leibnitz's Rule

Leibnitz's Rule : Let $f(x,t)$ be a C^1 function defined for $a \leq x \leq b$. Then

$$\frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b f_t(x,t)dx, \quad where \quad f_t(x,t) = \frac{\partial f}{\partial t}.$$

In other words, if we define $g(t) = \int_A^b f(x,t)dx$, then $\frac{dg}{dt} = \int_a^b f_t(x,t)dx$.

Leibnitz's rule also works for functions that have more parameters. For example, we have

$$\frac{d}{dt}\int_a^b f(x,y,z,t)dx = \int_a^b f_t(x,y,z,t)dx.$$

The rule can be generalized and applied to gradients. Indeed, let $f(x,y,z,t)$ be a scalar field defined for all $a \leq t \leq b$ and $(x,y,z) \in D$. Then if P is a point in D, we have that

$$\vec{\nabla}_P \int_a^b f dt = \int_A^b \vec{\nabla}_P f dt.$$

Let $f(x,y,z,t)$ be a time-dependent mass-density function on a region D. Then, by Leibnitz's rule, we have

$$\frac{d}{dt}\int\int\int_D f dV = \int\int\int_D \frac{\partial f}{\partial t} dV.$$

This means that the rate of change of the mass(the left-hand side of the equality) equals the integral of the rate of change of density(the right-hand side of the equality).

2 Formulas in Spherical Coordinates

The following are the formulas for gradient, divergence, and curl in spherical coordinates.

Let $g(\rho,\varphi,\theta)$ be a scalar field and let $\vec{G}(\rho,\varphi,\theta) = g_1(\rho,\varphi,\theta)\hat{\rho} + g_2(\rho,\varphi,\theta)\hat{\varphi} + g_3(\rho,\varphi,\theta)\hat{\theta}$ be a vector field, both in spherical coordinates. We have

$$\vec{\nabla} g = \frac{\partial g}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial g}{\partial \varphi}\hat{\varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial g}{\partial \theta}\hat{\theta}$$

$$div\vec{G} = \vec{\nabla}\cdot\vec{G} = \frac{1}{\rho^2}\frac{\partial(\rho^2 g_1)}{\partial \rho} + \frac{1}{\rho\sin\varphi}\frac{\partial(g_2 \sin\varphi)}{\partial\varphi} + \frac{1}{\rho\sin\varphi}\frac{\partial g_3}{\partial\theta}$$

$$\vec{curl}\vec{G} = \vec{\nabla}\times\vec{G} = \begin{vmatrix} \frac{1}{\rho^2\sin\varphi}\hat{\rho} & \frac{1}{\rho\sin\varphi}\hat{\varphi} & \frac{1}{\rho}\hat{\theta} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial\theta} \\ g_1 & \rho g_2 & \rho g_3\sin\varphi \end{vmatrix} =$$

$$= \left(\frac{1}{\rho\sin\varphi}\frac{\partial(g_3\sin\varphi)}{\partial\varphi} - \frac{1}{\rho\sin\varphi}\frac{\partial g_2}{\partial\theta}\right)\hat{\rho} +$$

$$+\left(\frac{1}{\rho\sin\varphi}\frac{\partial g_1}{\partial\theta} - \frac{1}{\rho}\frac{\partial(\rho g_3)}{\partial\rho}\right)\hat{\varphi} + \left(\frac{1}{\rho}\frac{\partial(\rho g_2)}{\partial\rho} - \frac{1}{\rho}\frac{\partial g_1}{\partial\varphi}\right)\hat{\theta}$$

3 The Laplacian

The Laplacian is a differential operator on scalar and vector fields. It is denoted by $\vec{\nabla}^2$. For a scalar field f, the Laplacian is defined in the following manner:

$$\vec{\nabla}^2 f = \vec{\nabla}\cdot\vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

For a vector field $\vec{F} = L\hat{\imath} + M\hat{\jmath} + N\hat{k}$, the operator is defined by

$$\vec{\nabla}^2\vec{F} = (\vec{\nabla}^2 L)\hat{\imath} + (\vec{\nabla}^2 M)\hat{\jmath} + (\vec{\nabla}^2 N)\hat{k}.$$

Definition 1 *A scalar field f is said to be* harmonic *if it satisfies the equation $\vec{\nabla}^2 f = 0$, called* Laplace's equation.

While for scalar fields we have $\vec{\nabla}^2 f = \vec{\nabla}\cdot\vec{\nabla} f$, for vector fields we can compute $\vec{\nabla}^2\vec{F}$ using the formula

$$\vec{\nabla}^2\vec{F} = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \vec{\nabla}\times(\vec{\nabla}\vec{F}).$$

Theorem 1 *Let $\vec{F},\vec{G}$ be differentiable vector fields on a simply connected domain D. Then $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{G}$ and $\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\vec{G}$ if and only if there exists a harmonic scalar field h such that $\vec{F} - \vec{G} = \vec{\nabla} h$.*

Theorem 1 can be used to prove the following result.

Theorem 2 (Helmholtz's Theorem) *Let* $\vec{F}$ *be differentiable on a simply connected domain* D*. There exist* $\vec{G}$ *and* $\vec{H}$ *such that:*

(i) $\vec{G}$ *is irrotational and* $\vec{H}$ *is divergenceless;*

(ii) $\vec{\nabla} \times (\vec{G} + \vec{H}) = \vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{F}$*;*

(iii) $\vec{\nabla} \cdot (\vec{G} + \vec{H}) = \vec{\nabla} \cdot \vec{G} = \vec{\nabla} \cdot \vec{F}$ *and*

(iv) $\vec{F} - (\vec{G} + \vec{H}) = \vec{\nabla} h$ *for some harmonic scalar field* h*.*