Lecture XXII

Surfaces

Recall that an elementary region $\hat{D}$ contained in $\mathbf{E^2}$ is called *convex* if for any points in $\hat{D}$, the line conecting them is contained in $\hat{D}$. Also recall that a map $\vec{R}$ on $\hat{D}$ is called *injective* if distinct points in $\hat{D}$ give different values of $\vec{R}$. Using these notions we will give a definition for surfaces.

Definition 1 (Parametric expression for a surface) *Let O be a fixed point in $\mathbf{E^3}$ and let $\hat{D}$ be a convex elementary region in $\mathbf{E^2}$, where $\mathbf{E^2}$ is the Cartesian plane of coordinates u, v. Let $\vec{R}(u, v)$ be an injective continuous function on $\hat{D}$ with values in $\mathbf{E^3}$. The* surface *S is the set of all points that have $\vec{R}(u, v)$ as position vector for some $(u, v) \in \hat{D}$.*

For example, in the uv plane we consider the rectangle $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and the function $\vec{R}(u, v) = \sin u \cos v\hat{i} + \sin u \sin v\hat{j} + \cos u\hat{k}$. Then the surface S that has parametric representation given by $\vec{R}$ is the sphere of radius 1 centered at O. Also, if we take $\hat{D}$ to be the disk of radius 1 centered at O, and $\vec{R}(u, v) = u\hat{i} + v\hat{j} + \sqrt{1 - u^2 - v^2}\hat{k}$ defined on $\hat{D}$, then the surface given by $\vec{R}$ is the sphere of radius 1 centered at O.

Definition 2 *Suppose $\vec{R}(u, v)$ has continuous partial derivatives. Let $\hat{P}$ be an interior point of $\hat{D}$. Then $\vec{w}(\hat{P}) = \frac{\partial \vec{R}}{\partial u}|_{\hat{P}} \times \frac{\partial \vec{R}}{\partial v}|_{\hat{P}}$ is called the* parametric normal vector *given by $\hat{P}$. Taking $P = \vec{R}(\hat{P})$, we can denote $\vec{w}(\hat{P})$ as $\vec{w}_P$ and call it the parametric normal vector at P on S.*

The parametric normal vector $\vec{w}_P$ has the following properties:

1. If $\vec{w}_P \neq 0$, then $\vec{w}_P$ is normal to S at P.
2. $|\vec{w}_P|$ is an amplification factor from the area in $\hat{D}$ to the surface area in S. More precisely, if $\hat{U}$ is a region around a point $\hat{P}$ in $\hat{D}$, and U is the corresponding region around P, then $surface\ area\ (U) \approx |\vec{w}_p|\ area\ (\hat{U})$.
3. The surface area of S is given by the double integral $\int\int_{\hat{D}} |\vec{w}(u, v)| dudv$.