Problem Set XI Solutions

1. $\oint$ 25.2 4efg.

(e)

$$\vec{F} = L\hat{i} + M\hat{j} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$$

$$L_x = (-y) \cdot 2x \cdot \frac{-1}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$M_y = x \cdot 2y \cdot \frac{-1}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$div\vec{F} = Lx + My = 0$$

(f)

$$\vec{F} = L\hat{i} + M\hat{j} + N\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$L_x = 1, \quad M_y = 1, \quad N_z = 1$$

$$div\vec{F} = L_x + M_y + N_z = 3$$

(g)

$$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$L_x = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + x \cdot 2x \cdot (-\frac{3}{2})\frac{1}{(x^2+y^2+z^2)^{\frac{5}{2}}} =$$

$$= \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

In the same manner we get

$$M_y = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \quad N_z = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Hence

$$div\vec{F} = L_x + M_y + N_z = 0$$

2. $\oint$ 25.2 5c,6b.

(5c)

$$\vec{F} = r^2\hat{r} = f(r)\hat{r}, \quad so \quad f(r) = r^2$$

$$div\vec{F} = \frac{1}{r}\frac{d(rf)}{dr} = \frac{1}{r}\frac{dr^2}{dr} = \frac{1}{r}3r^2 = 3r$$

(6b)

$$\vec{F} = \frac{1}{\rho^2}\hat{\rho} = f(\rho)\hat{\rho}, \quad so \quad f(\rho) = \frac{1}{\rho^2}$$

$$div\vec{F} = \frac{1}{\rho^2}\frac{d(\rho^2 f)}{d\rho} = \frac{1}{\rho^2}\frac{d1}{d\rho} = 0$$

3. $\oint$25.3 3.

(a)

$$\vec{F} = z\hat{k}, \quad so \quad div\vec{F} = 1$$

$$\int\int_S \vec{F}\cdot d\vec{\sigma} = \int\int\int_R div\vec{F}dV = \int\int\int_R 1dV = 12\pi$$

(b)

$$\vec{F} = -y\hat{\imath} + x\hat{\jmath}, \quad so \quad div\vec{F} = 0$$

$$\int\int_S \vec{F}\cdot d\vec{\sigma} = \int\int\int_R div\vec{F}dV = \int\int\int_R 0dV = 0$$

(c)

$$\vec{F} = x^3\hat{\imath}, \quad so \quad div\vec{F} = 3x^2$$

$$\int\int_S \vec{F}\cdot d\vec{\sigma} = \int\int\int_R div\vec{F}dV = \int\int\int_R 3x^2dV =$$

$$= 3\int_0^3\int_0^{2\pi}\int_0^2 \cos^2\theta r^3 drd\theta dz = 36\int_0^{2\pi}\cos^2\theta d\theta = 36\pi$$

4. $\oint$25.4 3bde.

(b) We have that $div\vec{F} = 3$, so

$$\oiint_S \vec{F}\cdot d\vec{\sigma} = \int\int\int_R div\vec{F}dV = 3\int\int\int_R 1dV = 3\pi a^2 b,$$

since R is the cylinder of radius a and altitude b, that has volume $\pi a^2 b$.

(d) Let S_1 be the triangle of vertices $(0,0,0)$, $(1,0,0)$, and $(0,0,1)$. A representation for S_1 is $\vec{R} = x\hat{\imath} + z\hat{k}$. Hence $\frac{d\vec{R}}{dx} = \hat{\imath}$ and $\frac{d\vec{R}}{dz} = \hat{k}$, so $\vec{w} = \frac{d\vec{R}}{dx} \times \frac{d\vec{R}}{dz} = -\hat{\jmath}$. Since in S_1, $y = 0$, then $\vec{F} = x\hat{\imath}$. Hence $\vec{F} \cdot \vec{w} = (x\hat{\imath}) \cdot (-\hat{\jmath}) = 0$. So

$$\int\int_{S_1} \vec{F} \cdot d\vec{\sigma} = 0$$

Let S_2 be the triangle of vertices $(0,0,0)$, $(0,1,0)$, and $(0,0,1)$. A representation for S_2 is $\vec{R} = y\hat{\jmath} + z\hat{k}$. Hence $\frac{d\vec{R}}{dy} = \hat{\jmath}$ and $\frac{d\vec{R}}{dz} = \hat{k}$, so $\vec{w} = \frac{d\vec{R}}{dy} \times \frac{d\vec{R}}{dz} = \hat{\imath}$. Since in S_2, $x = 0$, then $\vec{F} = y\hat{\jmath}$. Hence $\vec{F} \cdot \vec{w} = (y\hat{\jmath}) \cdot \hat{\imath} = 0$. So

$$\int\int_{S_2} \vec{F} \cdot d\vec{\sigma} = 0$$

Let S_3 be the triangle of vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,10$. A representation for S_3 is $\vec{R} = x\hat{\imath} + y\hat{\jmath}$. Hence $\frac{d\vec{R}}{dx} = \hat{\imath}$ and $\frac{d\vec{R}}{dy} = \hat{\jmath}$, so $\vec{w} = \frac{d\vec{R}}{dx} \times \frac{d\vec{R}}{dy} = \hat{k}$. Hence $\vec{F} \cdot \vec{w} = (x\hat{\imath} + y\hat{\jmath}) \cdot \hat{k} = 0$. So

$$\int_{S_3} \vec{F} \cdot d\vec{\sigma} = 0$$

Hence

$$\int\int_S \vec{F} \cdot d\vec{\sigma} = \oiint_{S+S_1+S_2+S_3} \vec{F} \cdot d\vec{\sigma} = -\int\int\int_R div\vec{F}dV,$$

where R is the region bounded by S, S_1, S_2, and S_3, since S is directed down, corresponding to an inward-directed region.

$$\int\int\int_R div\vec{F}dV = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 2dxdydz =$$

$$= 2\int_0 \int_0^{1-z} (1-y-z)dydz = 2\int_0^1 (1-z)^2 - \frac{(1-z)^2}{2}dz = \frac{1}{3}$$

Hence

$$\int\int_S \vec{F} \cdot d\vec{\sigma} = -\frac{1}{3}$$

(e) Let S_1 the top the cylinder and let S_2 be the bottom of the cylinder. Since $\vec{F}$ is directed parallel to S_1 and S_2, $\int\int_{S_1} \vec{F} \cdot d\vec{\sigma} = \int\int_{S_2} \vec{F} \cdot d\vec{\sigma} = 0$. Hence

$$\int\int_S \vec{F} \cdot d\vec{\sigma} = \int\int_{S+S_1+S_2} \vec{F} \cdot d\vec{\sigma}$$

By the divergence theorem,

$$\int\int_{S+S_1+S_2} \vec{F}\cdot d\vec{\sigma} = \int\int\int_R div\vec{F}dV = \int\int\int_R 1dV = 12\pi,$$

where R is the cylinder of radius 2 and altitude 3. Hence

$$\int\int_S \vec{F}\cdot d\vec{\sigma} = 12\pi$$

5. $\oint$25.5 3.

   (a)

$$\oiint_{S(B,\frac{1}{2})} \vec{F}\cdot d\vec{\sigma} = \oiint_{S(A,\frac{3}{2})} \vec{F}\cdot d\vec{\sigma} - \oiint_{S(A,\frac{1}{2})} \vec{F}\cdot d\vec{\sigma} = -4$$

   (b) If $S(P,a)$ contains neither A nor B, then $\oiint_{S(P,a)} \vec{F}\cdot d\vec{\sigma} = 0$. If $S(P,a)$ contains A but does not contain B, then $\oiint_{S(P,a)} \vec{F}\cdot d\vec{\sigma} = 1$. If $S(P,a)$ contains B but does not contain A, then $\oiint_{S(P,a)} \vec{F}\cdot d\vec{\sigma} = -4$. If $S(P,a)$ contains both A and B, then $\oiint_S(P,a)\vec{F}\cdot d\vec{\sigma} = -3$.

6. $\oint$26.3 1ac.

   (a)

$$\vec{F} = 2x\hat{i} - y\hat{j} - z\hat{k} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$L_y = L_z = M_x = M_z = N_x - N_y = 0$$

$$\vec{curl}\vec{F} = (N_y - M_z)\hat{i} + (L_z - N_x)\hat{j} + (M_x - L_y)\hat{k} = \vec{0}$$

   (c)

$$\vec{F} = yx\hat{i} + xz\hat{j} + xy\hat{k} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$L_y = x, L_z = 0, \quad M_x = z, M_z = x, \quad N_x = y, N_y = x$$

$$\vec{curl}\vec{F} = -y\hat{j} + (z-x)\hat{k}$$

7. $\oint$26.3 2ac.

   (a)

$$rot\vec{F}|_{\hat{u},P} = \vec{curl}\vec{F}\cdot\hat{u} = \vec{0}\cdot\hat{u} = 0$$

   (c)

$$rot\vec{F}|_{\hat{u},P} = \vec{curl}\vec{F}\cdot\hat{u} = \frac{2}{3}y + \frac{2}{3}(z-x) = \frac{2}{3}(y+z-x)$$

8. $\oint$26.4 3.

Directly:

$$\vec{curl F} = (z^2 + x)\hat{\imath} + (-z - 3)\hat{k}$$

The surface S can be parametrized by $\vec{R}(r, \theta) = r\cos\theta\hat{o} + r\sin\theta\hat{\jmath} + \frac{r^2}{2}\hat{k}$, with $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Then

$$\frac{d\vec{R}}{dr} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} + r\hat{k}, \quad \frac{d\vec{R}}{d\theta} = -r\sin\theta\hat{\imath} + r\cos\theta\hat{\jmath}, \quad so$$

$$w = \frac{d\vec{R}}{dr} \times \frac{d\vec{R}}{d\theta} = -r^2\cos\theta\hat{\imath} + r^2\sin\theta\hat{\jmath} + r\hat{k}.$$

Hence $\vec{F} = (\frac{r^4}{4} + r\cos\theta)\hat{\imath} + (-\frac{r^2}{2} - 3)\hat{k}$, so

$$\int\int_S \vec{curl F} \cdot d\sigma = \int_0^{2\pi}\int_0^2 (-\frac{r^6}{4}\cos\theta - r^3\cos^2\theta - \frac{r^3}{2} - 3r)drd\theta =$$

$$= \int_0^{2\pi}(-\frac{8}{7}\cos\theta - 4\cos^2\theta - 2 - 6)d\theta = -16\pi - 4\int_0^{2\pi}\cos^2\theta d\theta = -20\pi$$

By Stokes's Theorem,

$$\int\int_S \vec{curl F} \cdot d\vec{\sigma} = \oiint_C \vec{F} \cdot d\vec{R},$$

where C is a the circle of center $(0, 0, 2)$ and radius 2, and $\vec{R}(\theta) = 2\cos\theta\hat{\imath} + 2\sin\theta\hat{\jmath} + 2\hat{k}$. Then $\frac{d\vec{R}}{d\theta} = -2\sin\theta\hat{\imath} + 2\cos\theta\hat{\jmath}$ and $\vec{F}(\theta) = 6\sin\theta\hat{\imath} = 4\cos\theta\hat{\jmath} + 8\sin\theta\hat{k}$, so

$$\oiint_C \vec{F} \cdot d\vec{R} = \int_0^{2\pi} -12\sin^2\theta - 8\cos^2\theta d\theta =$$

$$= -16\pi - 4\int_0^{2\pi}\sin^2\theta d\theta = -20\pi$$

9. $\oint$26.4 6.

   (a) Consider the curve C_1 from $(3, 0)$ to $(-3, 0)$ with the path $\vec{R}(t) = t\hat{\imath}$, with $-3 \leq t \leq 3$. Then $\frac{d\vec{R}}{dt} = \hat{\imath}$ and $\vec{F} = -y\hat{\imath} + x\hat{\jmath} = t\hat{\jmath}$ for this path. Hence $\int_{C_1}\vec{F} \cdot d\vec{R} = \int_3^{-3} 0dt = 0$. Let S be the semicircle of radius 3 bounded by C and C_1. Then by Stokes's theorem:

$$\int_C \vec{F} \cdot d\vec{R} = \oint_{C+C_1}\vec{F} \cdot d\vec{R} = -\int\int_S \vec{curl F} \cdot d\vec{\sigma}$$

The representation for S is $\vec{R}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j}$, so $\vec{w} = \frac{d\vec{R}}{dr} \times \frac{d\vec{R}}{d\theta} = r\hat{k}$. Also, $\vec{F} = -y\hat{i} + x\hat{j}$, so $c\vec{url}\vec{F} = 2\hat{k}$. Hence

$$\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = \int_0^3 \int_0^\pi c\vec{url}\vec{F} \cdot \vec{w} d\theta dr =$$

$$= \int_0^3 \int_0^\pi 2r d\theta dr = 2\pi \int_0^3 r dr = 9\pi,$$

so

$$\int_C \vec{F} \cdot d\vec{R} = -\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = -9\pi$$

(b) Let C_3 be the curve from $(0,0)$ to $(-3,0)$ with the path $\vec{R}(t) = t\hat{i}$, $-3 \leq t \leq 0$. Then $\frac{d\vec{R}}{dt} = \hat{i}$ and $\vec{F} = -y\hat{i} + \hat{j} = t\hat{j}$ for this path. Hence $\int_{C_3} \vec{F} \cdot d\vec{R} = \int_0^{-3} 0dt = 0$. Let C_4 be the curve from $(3,3)$ to $(0,0)$ with the path $\vec{R}(t) = t\hat{i} + t\hat{j}$, $0 \leq t \leq 3$. Then $\frac{d\vec{R}}{dt} = \hat{i} + \hat{j}$ and $\vec{F} = -y\hat{i} + x\hat{j} = -t\hat{i} + t\hat{j}$ for this path. Hence $\int_{C_4} \vec{F} \cdot d\vec{R} = \int_3^0 0dt = 0$. Hence by Stokes's theorem,

$$\int_{C_1+C_2} \vec{F} \cdot d\vec{R} = \int_{C_1+C_2+C_3+C_4} \vec{F} \cdot d\vec{R} = -\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma},$$

where S is the surface bounded by C_1, C_2, C_3, and C_4. The representation for S is $\vec{R}(x,y) = (3 - y + x)\hat{i} + y\hat{j}$, with $0 \leq x \leq 3$ and $0 \leq y \leq 3$. Then $\vec{w} = \frac{d\vec{R}}{dx} \times \frac{d\vec{R}}{dy} = \hat{k}$ and since $c\vec{url}\vec{F} = 2\hat{k}$,

$$\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = \int_0^3 \int_0^3 \vec{F} \cdot \vec{w} dx dy = \int_0^3 \int_0^3 2dxdy = 18$$

Hence

$$\int_{C_1+C_2} = -\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = -18$$

10. $\oint$26.5 2.

Since $\vec{F} = x\hat{j}$, $c\vec{url}\vec{F} = \hat{k}$. We divide S into to surfaces S_1 and S_2. S_1 is the top of the cylinder and S_2 is the vertical part of the cylinder. Then

$$\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = \int\int_S \hat{k} \cdot d\vec{\sigma} = \int\int_{S_1} \hat{k} \cdot d\vec{\sigma} + \int\int_{S_2} \hat{k} \cdot d\vec{\sigma}$$

For S_1 a representation is $\vec{R} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + 3\hat{k}$. Then $\frac{d\vec{R}}{dr} = \cos\theta\hat{i} + \sin\theta\hat{j}$ and $\frac{d\vec{R}}{d\theta} = -r\sin\theta\hat{i} + r\cos\theta\hat{j}$, so $\vec{w} = \frac{d\vec{R}}{dr} \times \frac{d\vec{R}}{d\theta} = r\hat{k}$. Hence

$$\int\int_{S_1} \hat{k} \cdot d\vec{\sigma} = \int_0^2 \int_0^{2\pi} \hat{k} \cdot \vec{w} d\theta dr = \int_0^2 \int_0^{2\pi} r d\theta dr = 4\pi$$

For S_2 a representation is $\vec{R} = 2\cos\theta\hat{i} + 2\sin\theta\hat{j}$. Hence $\frac{d\vec{R}}{d\theta} = -2\sin\theta\hat{i} + 2\cos\theta\hat{j}$ and $\frac{d\vec{R}}{dz} = \hat{k}$, so $\vec{w} = \frac{d\vec{R}}{d\theta} \times \frac{d\vec{R}}{dz} = 2\cos\theta\hat{i} + 2\sin\theta\hat{j}$. Hence

$$\int\int_{S_1} \hat{k} \cdot d\vec{\sigma} = \int_0^2 \int_0^{2\pi} \hat{k} \cdot \vec{w} d\theta dr = 0$$

So

$$\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = \int\int_{S_1} \hat{k} \cdot d\vec{\sigma} + \int\int_{S_2} \hat{k} \cdot d\vec{\sigma} = 4\pi$$

The boundary of S is the circle C of center $(0,0,0)$ and radius 2. A path for C is $\vec{R} = 2\cos\theta\hat{i} + 2\sin\theta\hat{j}$. Then $\frac{d\vec{R}}{d\theta} = -2\sin\theta\hat{i} + 2\cos\theta\hat{j}$ with $0 \le \theta \le 2\pi$. Also, $\vec{F} = x\hat{j} = 2\cos\theta\hat{j}$. Hence $\vec{F} \cdot \frac{d\vec{R}}{d\theta} = 4\cos^2\theta$. Then

$$\oint_C \vec{F} \cdot d\vec{R} = \int_0^{2\pi} \vec{F} \cdot \frac{d\vec{R}}{d\theta} d\theta = \int_0^{2\pi} 4\cos^2\theta d\theta = 4\pi$$

This confirms Stokes's theorem:

$$\int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = \oint_C \vec{F} \cdot d\vec{R} = 4\pi$$

11. $\oint$26.6 2.

(a)

$$\vec{F} = 2x\hat{i} - y\hat{j} - z\hat{k} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$L_y = L_z = M_x = M_z = N_x - N_y = 0$$

$$c\vec{url}\vec{F} = (N_y - M_z)\hat{i} + (L_z - N_x)\hat{j} + (M_x - L_y)\hat{k} = \vec{0}$$

Hence $\vec{F}$ is conservative.

(b)

$$\vec{F} = x^2\hat{i} + yz\hat{j} - yz\hat{k} = l\hat{i} + M\hat{j} + N\hat{k}$$

$$L_y = L_z = 0, \quad Mx = 0, \quad M_z = y, \quad N_x = 0, \quad N_y = z$$

$$c\vec{url}\vec{F} = (-z - y)\hat{i} \neq \vec{0}$$

Hence $\vec{F}$ is not conservative.

(c)

$$\vec{F} = yx\hat{i} + xz\hat{j} + xy\hat{k} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$L_y = x, L_z = 0, \quad M_x = z, M_z = x, \quad N_x = y, N_y = x$$

$$c\vec{url}\vec{F} = -y\hat{j} + (z - x)\hat{k} \neq \vec{0}$$

Hence $\vec{F}$ is not conservative.

(d)

$$\vec{F} = y^2\hat{\imath} + z^2\hat{\jmath} + x^2\hat{k}$$

$$L_y = 2y, L_z = 0, \quad M_x = 0, M_z = 2z, \quad N_x = 2x, N_y = 0$$

$$c\vec{url}\vec{F} = -2z\hat{\imath} - 2x\hat{\jmath} - 2y\hat{k} \neq \vec{0}$$

Hence $\vec{F}$ is not conservative.

12. $\oint$26.5 6.

(a) Let S be a cylinder of basis C_1 and infinite altitude such that the axis y is the axis of the cylinder. Then the boundary of S is C_1. So by Stokes's theorem,

$$\oint_{C_1} \vec{F} \cdot d\vec{R} = \int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = 0$$

(b) Let S be a cylinder with C_3 its bottom and C_2 its top, such that the circle $x^2 + y^2 = 1$ in the plane $z = 0$ is exterior to the cylinder. Then the boundary of S is formed by C_2 and C_3. So by Stokes's theorem,

$$\oint_{C_2} \vec{F} \cdot d\vec{R} - \oint_{C_3} \vec{F} \cdot d\vec{R} = \int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = 0,$$

$$so \quad \oint_{C_2} \vec{F} \cdot d\vec{R} = \oint_{C_3} \vec{F} \cdot d\vec{R},$$

since C_2 and C_3 have opposite directions.

(c) Imagine a sphere with three holes C_2, C_4 and C_5. Let the line z enter the sphere through C_2 and exit the sphere through C_5. Let the circle $x^2 + y^2 = 1$, $z = 0$ enter the sphere through C_4 and exit the sphere through C_5. Then the sphere S is the surface we are searching. By Stokes's theorem,

$$\oint_{C_2} \vec{F} \cdot d\vec{R} - \oint_{C_4} \vec{F} \cdot d\vec{R} - \oint_{C_5} \vec{F} \cdot d\vec{R} = \int\int_S c\vec{url}\vec{F} \cdot d\vec{\sigma} = 0,$$

$$so \quad \oint_{C_2} \vec{F} \cdot d\vec{R} = \oint_{C_4} \vec{F} \cdot d\vec{R} + \oint_{C_5} \vec{F} \cdot d\vec{R},$$

since C_2 has counterclockwise direction, while C_4 and C_5 have clockwise direction.

(d) Suppose there is a surface S which lies entirely in D and has C_1, C_4 and C_5 as its entire boundary. Then

$$\int\int_S c\vec{url}\vec{F}\cdot d\vec{\sigma} = \pm\oint_{C_1}\vec{F}\cdot d\vec{R} \pm \oint_{C_4}\vec{F}\cdot d\vec{R} \pm \oint_{C_5}\vec{F}\cdot d\vec{R} = 0$$

But from (a), we have that $\oint_{C_1}\vec{F}\cdot d\vec{R} = 0$, and from the magnetic field we have that $\oint_{C_5}\vec{F}\cdot d\vec{R} = 0$. Hence $\oint_{C_4}\vec{F}\cdot d\vec{R} = 0$, which is false, since the circulation of $\vec{F}$ on C_4 is not 0. Hence there is no surface S that lies entirely in D and C_1, C_4 and C_5 as its entire boundary.