Lecture XXVI

The Divergence Theorem

In this lecture, we will define a new type of derivative for vector fields on $\mathbf{E^3}$, called *divergence*. Let $\vec{F}$ be a vector field defined on a domain D. Let us start by defining the divergence of $\vec{F}$ on interior points of D, i.e. points P such that there exists a sphere of center P and radius $a > 0$ with its interior contained in D.

Definition 1 *Let $\vec{F}$ be a continuous vector field on D in $\mathbf{E^3}$. Let P be an interior point of D, and let $S(P,a)$ be the sphere of center P and radius a, for all $a > 0$. The volume of $S(P,a)$ is $\frac{4\pi a^3}{3}$ and the flux of $\vec{F}$ through $S(P,a)$ is $\oiint_{S(P,a)} \vec{F} \cdot d\vec{\sigma}$. Consider the limit*

$$\lim_{a \to 0} \frac{3}{4\pi a^3} \oiint_{S(P,a)} \vec{F} \cdot d\vec{\sigma}.$$

If this limit exists, then it is called the divergence of $\vec{F}$ at P, *and it is denoted by $div\vec{F}|_P$.*

Below are two important properties of divergence.

1. Existence: If $\vec{F}$ is C^1 on D, then the divergence of $\vec{F}$ exists at every interior point of D.

2. Linearity: If $\vec{F}$ and $\vec{G}$ are vector fields defined on D, for any two scalar constants a and b the following equality holds:

$$div(a\vec{F} + b\vec{G}) = a(div\vec{F}) + b(div\vec{G}).$$

The following theorem helps us find a formula for divergence in Cartesian coordinates.

Theorem 1 (The parallel flow theorem in $\mathbf{E^3}$) *Let $\vec{F}$ be a vector field on D in $\mathbf{E^3}$ such that there exists a scalar field on D and a constant unit vector $\hat{w}$ for which $\vec{F} = f\hat{w}$ on D. Such a vector field $\vec{F}$ is called a* parallel flow*. Suppose f is C^1. Then for any interior point P of D, the following equality holds:*

$$div\vec{F}\big|_P = \frac{df}{ds}\Big|_{\hat{w},P}.$$

Let $\vec{F} = L\hat{\imath} + M\hat{\jmath} + N\hat{k}$ be a C^1 vector field on a domain D in $\mathbf{E^3}$. By using the linearity of divergence and applying the parallel flow theorem to $L\hat{\imath}$, $M\hat{\jmath}$, and $N\hat{k}$, we get the following formula:

$$div\vec{F}\big|_P = L_x(P) + M_y(P) + N_z(P) = \frac{\partial L}{\partial x}\Big|_P + \frac{\partial M}{\partial y}\Big|_P + \frac{\partial N}{\partial z}\Big|_P$$

Through the following theorem, we can use divergence to compute surface integrals more easily.

Theorem 2 (The divergence theorem) *Let $\vec{F}$ be a C^1 vector field on D in $\mathbf{E^3}$. Let R be a regular region in D, with outward directed outer boundary surface S and inward directed inner boundary surfaces $S_1, \ldots, S_n$. Then*

$$\int\int\int_R div\vec{F}dV = \oiint_S \vec{F}\cdot d\vec{\sigma} + \oiint_{S_1} \vec{F}\cdot d\vec{\sigma} + \cdots + \oiint_{S_n} \vec{F}\cdot d\vec{\sigma}.$$

Let R be a regular region in D, and let S be its boundary. Then, by the divergence theorem, we have that

$$\int\int\int_R div\vec{F}dV = \oiint_S \vec{F}\cdot d\vec{\sigma}.$$