18.022 Hour Test

November- 29, 2005

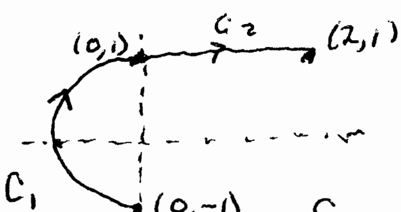
CLOSED BOOK; NO BOOKS, NOTES, OR CALCULATORS SOLUTIONS

Name ______________ Rec.Instr. ______________ Rec.Time ______________

Please show all your work on this paper, and identify your answers clearly. Use backs of pages if necessary. Points for each question are as shown (for a total of 100 points). If you have difficulty on a problem, go on to the next.

1. (20) The directed path C in the xy plane consists of two parts: first, a left semicircle from (0,-1) to (0,1), with center at the origin; and then, second, a straight segment from (0,1) to (2,1). Let $\vec{F} = -y\hat{i} + x\hat{j}$. Find $\int_C \vec{F}\cdot d\vec{R}$. (Be careful about path-direction vs. parameter-direction.)

Answer: $\boxed{-(\pi + 2)}$



$$\int_C = \int_{C_1} + \int_{C_2}$$

C_1: $\vec{R} = \hat{i}\cos t + \hat{j}\sin t$ $\quad (\frac{\pi}{2} \le t \le \frac{3\pi}{2})$

(direction opp. to parameter)

$$\frac{d\vec{R}}{dt} = -\hat{i}\sin t + \hat{j}\cos t$$

$$\vec{F} = -y\hat{i} + x\hat{j} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\therefore \int_{C_1} = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\vec{F}\cdot\frac{d\vec{R}}{dt}\right)dt$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\sin^2 t + \cos^2 t)\,dt$$

$$= -\pi$$

C_2: $\vec{R} = \hat{i}t + \hat{j}$ $\quad (0 \le t \le 2)$

$$\frac{d\vec{R}}{dt} = \hat{i}$$

$$\vec{F} = -y\hat{i} + x\hat{j} = -\hat{i} + t\hat{j}$$

$$\therefore \int_{C_2} = \int_0^2 \left(\vec{F}\cdot\frac{d\vec{R}}{dt}\right)dt = \int_0^2 (-1)\,dt$$

$$\therefore \int_C = \int_{C_1} + \int_{C_2} = -(\pi + 2)$$

SCORE

1.

2.

3.

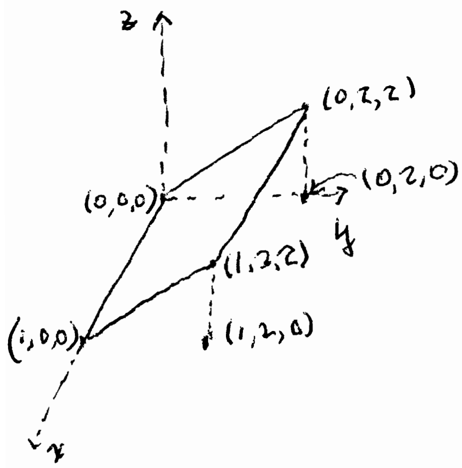
4.

5.

TOTAL:

2. (20) Let S be the rectangular surface with vertices at (0,0,0), (1,0,0), (1,2,2), and (0,2,2). Let S be directed upward. Let $\vec{F} = y^2\hat{i} - z^2\hat{j} + x^2\hat{k}$. Find the flux of $\vec{F}$ through S.

Answer: $\boxed{\dfrac{10}{3}}$



S is graph of $z = f(x,y) = y$

$$\vec{w} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} = -\hat{j} + \hat{k}$$

On S, as function of x and y, $\vec{F} = y^2\hat{i} - z^2\hat{j} + x^2\hat{k}$

$$= y^2\hat{i} - y^2\hat{j} + x^2\hat{k}$$

$$\therefore \vec{F}\cdot\vec{w} = y^2 + x^2$$

$$\therefore \iint_S = \int_0^2\int_0^1 (x^2+y^2)\,dx\,dy = \int_0^2 \left(\frac{x^3}{3} + y^2x\right)\Big]_0^1 dy = \int_0^2\left(\frac{1}{3}+y^2\right)dy$$

$$= \frac{y+y^3}{3}\Big]_0^2 = \frac{10}{3}.$$

3. (20) Let the surface S be the upper hemisphere of radius 1 with center at the origin. S is directed upward. Let $\vec{F} = x\hat{i} - y\hat{j} + z\hat{k}$. Find the flux of $\vec{F}$ through S by introducing a circular disk S' in the xy plane so that S and S' form a closed surface and by then applying the divergence theorem to $\vec{F}$ over the hemispherical solid region enclosed by S and S'.

$\vec{\nabla}\cdot\vec{F} = (1 - 1 + 1) = 1$

Answer: $\boxed{\dfrac{2\pi}{3}}$

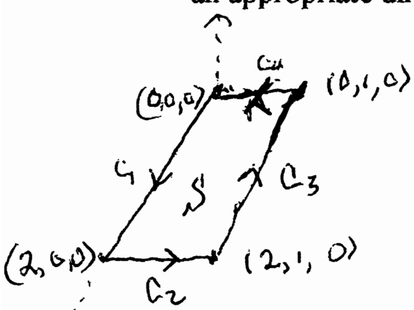
$\therefore \iiint_R \vec{\nabla}\cdot\vec{F}\,dV = 1\,(\text{volume of } R) = \frac{2\pi}{3}$

By divergence theorem, $\frac{2\pi}{3} = \oiint_{S+S'} \vec{F}\cdot d\vec{\sigma}$

But $\iint_{S'} \vec{F}\cdot d\vec{\sigma} = \iint_{S'} (\vec{F}\cdot\hat{k})\,d\sigma = 0$, since $z = 0$ on S'

$\therefore \iint_{S'} \vec{F}\cdot d\vec{\sigma} = 0$ and $\iint_S \vec{F}\cdot d\vec{\sigma} = \frac{2\pi}{3}$.

4. (20) Let S be the rectangle in the xy plane with vertices (0,0,0), (2,0,0), (2,1,0), and (0,1,0). S is directed up. Let $\vec{F} = z^2\hat{i} + x^2\hat{j} + y^2\hat{k}$. Let C be the boundary of S. Choose an appropriate direction for C, and verify Stokes's theorem for $\vec{F}$, S, and C.



Answer: $\iint_S = \boxed{4}$ $\oint_C = \boxed{4}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = \hat{i}(2y) + \hat{j}(2z) + \hat{k}(2x)$$

On S, $\vec{\nabla}\times\vec{F}\cdot\hat{n} = (\vec{\nabla}\times\vec{F})\cdot\hat{k} = 2x$

$$\therefore \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{\sigma} = \int_0^1\int_0^2 2x\,dx\,dy = \int_0^1 4\,dy = 4$$

On $C_1 + C_2 + C_3 + C_4$:

$\int_{C_1}$: $\vec{F}\cdot\hat{T} = z^2 = 0 \Rightarrow \int_{C_1} = 0$

$\int_{C_2}$: $\vec{F}\cdot\hat{T} = x^2 = 4 = \int_{C_2} = \int_0^1 4\,dt = 4$

$\int_{C_3}$: $\vec{F}\cdot\hat{T} = -z^2 = 0 \Rightarrow \int_{C_3} = 0$

$\int_{C_4}$ $\vec{F}\cdot\hat{T} = x^2 = 0 \Rightarrow \int_{C_4} = 0$

$\therefore \oint_C \vec{F}\cdot d\vec{R} = 4.$

5. (20) $\vec{F}$ is a divergenceless vector field whose domain is all of space except for the two points
A = (0,0,0) and B = (0,1,0). For any point P, let S(P,a) be the outward directed sphere with radius a and center at P. You are given that the flux of $\vec{F}$ through S(A,1/2) is 1 and through S(A,3/2) is -3.

(a) What is the flux of $\vec{F}$ through S(B,1/2)?

(b) What are the possible values that can occur for the flux of $\vec{F}$ through S(P,a) as P and a vary?

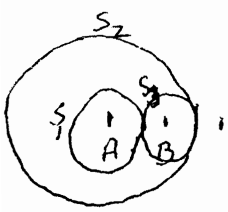
Answer: (a): $\boxed{-4}$

(b): $\boxed{0, 1, -4, -3}$

(a) Let $S_1 = S(A, 1/2)$

$S_2 = S(A, 3/2)$

$S_3 = S(B, 1/2)$



By divergence theorem: $\iint_{S_2} - \iint_{S_1} - \iint_{S_3} = 0$

$\Rightarrow -3 - 1 - \iint_{S_3} = 0 \Rightarrow \iint_{S_3} = -4$

(b) A sphere (surface) which lies in domain must contain: neither A nor B, A but not B, B but not A, both A and B

By divergence theorem, possible values are 0, 1, -4, -3

Footnote (unrelated to today's test): In the recent "practice problems", the correct answer to 8e is "T".