

18.022 Fall 05

Practice Problems for Hour Test 3 with Answers

1. Evaluate each of the following line integrals. In each case, use the gradient-field theorem if you can; otherwise, use parametric evaluation. In each case, the directed curve C consists of the straight segment from $(0,0)$ to $(1,1)$ followed by the straight segment from $(1,1)$ to $(0,2)$.

(a) $\int_C \vec{F} \cdot d\vec{R}$, where $\vec{F}(x, y) = xy^2\hat{i} + (x^2y + y)\hat{j}$. $\mathbf{f} = \frac{1}{2}(x^2y^2 + y^2)$. **VLI = 2.**

(b) $\int_C \vec{G} \cdot d\vec{R}$, where $\vec{G}(x, y) = xy\hat{i} + xy\hat{j}$. **VLI = 2/3**

(Hint. Note that a straight segment from (a,b) to (c,d) can be parameterized by $x(t) = a + t(c - a)$ and $y(t) = b + t(d - b)$, for $0 \leq t \leq 1$.)

2. For each statement, put “T” if the statement is true and “F” if the statement is sometimes false. **Each correct answer counts +4. Each incorrect answer counts -2. Each answer left blank counts 0.**

(a) F $\iiint_R x \, dV$ gives the coordinate \bar{x} of the centroid of the region R .

(b) T If a scalar field f is continuous on the compact domain R , it follows that every proper sequence of Riemann sums for the multiple integral $\iiint_R f \, dV$ must have the same unique limit.

(c) F The iterated integrals $\int_0^1 \int_{x^2}^x dy dx$ and $\int_0^1 \int_{y^2}^y dx dy$ are integrals over the same region in the xy plane.

(d) T $\vec{F} = (1/r)\hat{r}$ has independence of path.

(e) T $\vec{F} = r\hat{r}$ is a gradient field.

3. A circular cone has as its base a circular disk in the xy plane with radius 4 and center at the origin. Its vertex is at the point $(0, 0, 4)$. Let S be that portion of the surface of the cone which lies between the planes $z = 1$ and $z = 3$.

(a) Using the cylindrical coordinates r and θ as parameters, give a parameterization $\vec{R}(r, \theta) = x(r, \theta)\hat{i} + y(r, \theta)\hat{j} + z(r, \theta)\hat{k}$ for the surface S .

Ans: $r\cos\theta\hat{i} + r\sin\theta\hat{j} + (4 - r)\hat{k}$ for $1 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$

(b) Find $\frac{\partial \vec{R}}{\partial r}$ and $\frac{\partial \vec{R}}{\partial \theta}$; find the parametric normal vector \vec{w} ; and find $|\vec{w}|$. (Find these as vector or scalar functions of the parameters r and θ .)

Ans: $\cos\theta\hat{i} + \sin\theta\hat{j} - \hat{k}$; $-\text{rsin}\theta\hat{i} + \text{rcos}\theta\hat{j}$; $\vec{w} = \text{rcos}\theta\hat{i} + \text{rsin}\theta\hat{j} + r\hat{k}$; $|\vec{w}| = r\sqrt{2}$

(c) Set up and evaluate a double integral in r and θ for $\iint_S d\sigma =$ surface area of S .

$$8\sqrt{2}\pi$$

4. (a) State Green's Theorem for a vector field \vec{F} in E^2 and a regular region R with the outside boundary curve C and the interior boundary curve C_1 .

(b) Verify Green's Theorem for the case where $\vec{F} = -y\hat{i} + x\hat{j}$ and R is the region contained between the circle C of radius 3 with center at the origin and the circle C_1 of radius 1 with center at the origin.

(b) $\iint_R \text{rot } \vec{F} dA = 16\pi$ and $\oint_C + \oint_{C_1} = 16\pi$

5. Let \vec{F} be an irrotational field defined on all points in E^2 except the points $P = (1,0)$ and $Q = (0,1)$. Let C_1 be a counterclockwise circle of radius 2 with center P , and let C_2 be a clockwise circle of radius 1 with center Q . Assume that $\oint_{C_1} \vec{F} \cdot d\vec{R} = 5$ and

$\oint_{C_2} \vec{F} \cdot d\vec{R} = 3$. Find all possible values of $\oint_C \vec{F} \cdot d\vec{R}$ for an arbitrarily given simple closed curve C in the domain of \vec{F} .

$$0, \pm 3, \pm 5, \pm 8$$

6. One or both of the following vector fields on E^3 is conservative. Identify each conservative field and find a potential for it.

(a) $\vec{F} = (y+3)\hat{i} + (x+2z)\hat{j} + (2y-1)\hat{k}$. **Cons. $f(x,y) = xy+2yz+3x-z$**

(b) $\vec{G} = 3y\hat{i} + (3x+3z)\hat{j} + 4y\hat{k}$ **not cons**

7. Let $\vec{F} = 5y\hat{i} + 3x\hat{j}$ on the xy plane. Use Green's theorem to calculate the circulation measure for \vec{F} on the region R , where R has the circle of radius 3 with center at the origin as its outer boundary and the circle of radius 1 with center at $(1,0)$ as its inner boundary.

$$-16\pi$$

8. True or false as in 2:

(a) **F** Let $\vec{F} = (2y-2x)\hat{i} + (2x+2y)\hat{j}$. Let $P = (0,0)$ and $Q = (1,2)$. Let C be the directed line segment from P to Q . Then $\int_C \vec{F} \cdot d\vec{R} = 4$.

(b) F $\vec{F} = (2x - 2y)\hat{i} + (2x + 2y)\hat{j}$ is a conservative field.

(c) T $\vec{F} = (1/r)\hat{r}$ has independence of path.

(d) T $\vec{F} = r\hat{r}$ is a gradient field.

(e) F Let $\vec{F} = (1/r)\hat{r}$. Let S be the open-ended cylinder of radius 2, between $z = 0$ and $z = 3$, with the z axis as its axis. S is directed out. Then $\iint_S \vec{F} \cdot d\vec{\sigma} = 6\pi$

9. As in 2:

Assume that \vec{F} is a C^1 vector field defined on a simply connected domain. Then:

(a) T If \vec{F} is irrotational, then \vec{F} is conservative.

(b) T If \vec{F} is conservative, then \vec{F} is irrotational.

(c) F If $\vec{F} = r\hat{r}$, then \vec{F} is divergenceless.

(d) T If $\vec{F} = r\hat{r}$, then \vec{F} is irrotational.

(e) T If $\vec{F} = \vec{\nabla} \times \vec{G}$, then \vec{F} is divergenceless.

10. Find the value of $\oiint_S \vec{F} \cdot d\vec{\sigma}$, where $\vec{F} = 3xz\hat{i} + 2y\hat{j} - 3z\hat{k}$, and S is the outward-directed closed cylinder (top and bottom included) of radius 2 which has the z axis as its axis, its top in the plane $z = 3$, and its bottom in the plane $z = 0$. (Use the divergence theorem.)

42π

11. Let $\vec{F} = (2xy + 3)\hat{i} + (x^2 - 2)\hat{j}$ on the xy plane. Let C be the portion of the ellipse $(x^2/4) + (y^2/25) = 1$ which lies in the first quadrant and is directed from left to right. Find $\int_C \vec{F} \cdot d\vec{R}$. (Try to avoid a parametric calculation.)

16; F is conservative; use potential