Lecture X

Linear Approximation

Chain Rule

1 Linear Approximation; Gradient

We say that a function has a linear approximation on a domain D if it has a linear approximation at any point $P \in D$.

Theorem 1 *If f has a linear approximation on a domain D, then f is continuous on D.*

Theorem 2 *If f has continuous partial derivatives on D, then f has linear approximation on D.*

Definition 1 *Let f be a function with partial derivatives at P. The vector*

$$\vec{G}_P = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

is called the gradient at P *and is usually denoted by* $\vec{\nabla} f|_P$.

Theorem 3 (Gradient Theorem) *Let f have domain D in $\mathbf{E^3}$ and let P be a point in D. If $\hat{u}$ is a fixed unit vector, then*

$$\frac{df}{ds}\Big|_{\hat{u},P} = \frac{\partial f}{\partial x}\Big|_P \hat{i} + \frac{\partial f}{\partial y}\Big|_P \hat{j} + \frac{\partial f}{\partial z}\Big|_P \hat{k} = \vec{\nabla} f|_P \cdot \hat{u}$$

2 Chain Rule

Physical variables, quantities that can be measured in a given physical system, are related in various ways. For example, pressure p is a function of volume V and temperature T, $p = h(V, T)$. Also, T depends on V and internal energy U, $T = t(V, U)$. From these to equalities, we get that p is a function of volume V

and internal energy U, $p = h(V,T) = h(V,t(V,U)) = g(V,U)$. So to find the partial derivative of p with respect to V, we have to specifiy the function we are referring to: h or g. We write $(\frac{\partial p}{\partial V})_U = \frac{\partial g}{\partial V}$ and $(\frac{\partial p}{\partial V})_T = \frac{\partial h}{\partial V}$. Below we will see how such partial derivatives are related. By $(\frac{\partial f}{\partial x})_y$ we mean the partial derivative of f with respect to x when f is seen as a function of variables x and y.

Theorem 4 (Chain Rule) *Let w, u, v, x, y be physical variables such that there exist differentiable functions f, g, h such that $w = f(u,v)$, $u = g(x,y)$ and $v = h(x,y)$. Then*

$$\left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial f}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial h}{\partial x}$$

$$\left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial f}{\partial u}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial h}{\partial y}$$

If we take $w = f(u,v)$, $u = g(x,y)$, and $v = h(x)$, then applying the Chain Rule we get

$$\left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial f}{\partial u}\frac{\partial g}{\partial y},$$

since $\frac{\partial h}{\partial y} = 0$.