Lecture III
Vector Algebra in Cartesian Coordinates

Let us construct a Cartesian coordinates system in $\mathbf{E^3}$. First we choose a point O, called the *origin*. Then we chose three mutually perpendicular rays starting from O. These rays are called the *positive x axis, positive y axis,* and *positive z axis*. Consider the lines containing these rays. For any of these lines, every point on it can be identified with a real number: if the point is on the ray, the real number is the distance to O, if it's not on the ray, the number is the distance to O times -1. Let us denote these three lines by X, Y, and Z. Let P be a point in space. Consider the projection-points of P on X, Y, and Z. These points give the Cartesian coordinates of P, denoted x_P, y_P, and z_P. Any triplet of real numbers forms the cordinatesfor some point P. Different points have different coordinates.

The three unit vectors in the directions of the positive x, y, and z axes are customarily denoted by $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$. Let $\vec{A}$ be a vector in $\mathbf{E^3}$ and let P be the point such that $\vec{OP} = \vec{A}$. Let (a_1, a_2, a_3) be the coordinates of P. Consider the vectors $\vec{A_1} = a_1\hat{\imath}$, $\vec{A_2} = a_2\hat{\jmath}$, and $\vec{A_3} = a_3\hat{k}$. by vector addition and multiplication with scalars, one obtains the following expression:

$$\vec{A} = \vec{A_1} + \vec{A_2} + \vec{A_3} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

Then a_1, a_2, and a_3 are called the *scalar components* of $\vec{A}$, and $a_1\hat{\imath}, a_2\hat{\jmath}$, and $a_3\hat{k}$ are called the *vector components* of $\vec{A}$. By using the $\hat{\imath}, \hat{\jmath}, \hat{k}$ unit vectors, we obtain coordinate formulas for the four basic vector operations:

1. Multiplication by a scalar

   $$c\vec{A} = (ca_1)\hat{\imath} + (ca_2)\hat{\jmath} + (ca_3)\hat{k},$$

   for any scalar c.

2. Addition of vectors

If $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{A} + \vec{B} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}.$$

3. Dot product

If $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, considering that $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = \hat{\imath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\jmath} = 0$, we get that

$$\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3.$$

4. Cross product

Let $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$. To compute the cross product, we use the following equalities:

$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0, \qquad \text{and}$$

$$\hat{\imath}\times\hat{\jmath} = -\hat{\jmath}\times\hat{\imath} = \hat{k}, \hat{\jmath}\times\hat{k} = -\hat{k}\times\hat{\jmath} = \hat{\imath}, \hat{k}\times\hat{\imath} = -\hat{\imath}\times\hat{k} = \hat{\jmath}.$$

We get the following formula:

$$\vec{A}\times\vec{B} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}.$$

Using determinants one can easily remember the formula, since:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Considering that $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$, we get that $\vec{A}$ can be expressed as

$$\vec{A} = (\vec{A}\cdot\hat{\imath})\hat{\imath} + (\vec{A}\cdot\hat{\jmath})\hat{\jmath} + (\vec{A}\cdot\hat{k})\hat{k},$$

which is known as the *frame identity*.

Triple products

1. Scalar triple product

Let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,and $\vec{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. The triple product $(\vec{A} \times \vec{B})\cdot\vec{C}$ is called a scalar triple product, since it is a scalar quantity. Its value is given by:

$$(\vec{A} \times \vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B} \times \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Hence we can simply write $(\vec{A}\times\vec{B})\cdot\vec{C}$ as $[\vec{A},\ \vec{B},\vec{C}]$ without specifying the positions for the cross and dot signs. We will use the following equalities in computing quadruple products:

$$[\vec{A},\vec{B},\vec{C}] = [\vec{B},\vec{C},\vec{A}] = [\vec{C},\vec{A},\vec{B}] = -[\vec{A},\vec{C},\vec{B}] = -[\vec{C},\vec{B},\vec{A}] = -[\vec{B},\vec{A},\vec{C}].$$

2. Vector triple product

   The cross product is not associative so we will give two formulas:

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{C}\cdot\vec{A})\vec{B} - (\vec{C}\cdot\vec{B})\vec{A}, \qquad \text{and}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Quadruple products

1. Scalar quadruple product

   The expression $(\vec{A} \times \vec{B})\cdot(\vec{C} \times \vec{D})$ is called a quadruple scalar product, and by applying the formulas for triple products, we get the value:

$$(\vec{A} \times \vec{B})\cdot(\vec{C} \times \vec{D}) = (\vec{C}\cdot\vec{A})(\vec{B}\cdot\vec{D}) - (\vec{C}\cdot\vec{B})(\vec{A}\cdot\vec{D})$$

2. Vector quadruple product

   The expression $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$ is called a quadruple vector product, and by applying the formulas for triple products, we get the value:

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{C},\vec{D},\vec{A}]\vec{B} - [\vec{C},\vec{D},\vec{B}]\vec{A}$$